Multiplicity of Equilibria in Search Markets with Free Entry and Exit

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Abstract: In this paper I identify an instance of multiplicity of equilibria, in search markets with production cost heterogeneity. I show that if firms may enter or exit the market, there may be multiple equilibria. I provide a monotonicity property, which is necessary and sufficient for uniqueness. Multiplicity vanishes as the search cost gets small. Multiple equilibria can be ranked by welfare.

Key Words: *Search, Multiplicity of Equilibria, Coordination Failure*
JEL Classification: *D43, D83, L13*

1 Introduction

In this paper I identify an instance of multiplicity of equilibria in search markets. The analysis applies to a class of partial equilibrium search models, related to Benabou (1993), MacMinn (1980), and Reinganum (1979). In the model, firms may enter or exit the market, have different marginal costs, and set prices. Consumers search for prices.

Costly search leads consumers to accept prices above the minimum charged in the market. Low cost firms charge the lowest prices. Since consumers accept prices above the minimum charged in the market, low cost firms are not constrained by consumer search, and charge their monopoly price. If the consumers' reservation price is high, high cost firms charge the reservation price. If the reservation price is low, high cost firms exit the industry.

The consumers' reservation price depends on the consumers' beliefs about the price distribution, and affects the measure of active firms. The measure of active firms affects the price distribution. In equilibrium, the consumers' beliefs about the price distribution are correct. Thus, the measure of active firms affects the reservation price. On the one hand, a higher reservation price allows higher cost firms to be active, i.e., to enter, or not to leave the industry, depending on the interpretation. On the other hand, if

higher cost firms are active, consumers expect to face a worse price distribution, and hold a higher reservation price. Since the reservation price and the measure of active firms are mutually reinforcing, different levels of the reservation price may arise as equilibria.

I provide a necessary and sufficient condition for uniqueness, which consists of a monotonicity property. The condition holds trivially in Reinganum (1979), but need not hold in general. Multiplicity vanishes, as the search cost gets small.

When multiplicity occurs, equilibria can be ranked by welfare. Lower reservation prices induce high cost firms to be inactive, and thereby redistribute production from high to low cost firms. There is a potential coordination failure, since no mechanism ensures that the industry will be at the equilibrium with the lowest reservation price.

In Section 2 I present the model, and in Section 3 I characterize its equilibria. In Section 4 I discuss multiplicity, in Section 5 I discuss extensions, and in Section 6 I discuss welfare analysis. Section 7 concludes.

2 The Model

In this section I present the model.

2.1 The Setting

Consider a market for a homogeneous search good that opens for 1 period. The game consists of 2 stages. In *stage* 1 firms choose prices, and in *stage* 2 consumers search for prices.

2.2 Consumers

There is a unit measure continuum of risk neutral identical consumers. A consumer who buys at price p on $\Re_0^+$ demands $D(p)$, where $D(.)$ is a twice differentiable, bounded function, with a bounded inverse, and strictly decreasing for p on $[0, D^{-1}(0)]$. Denote by $S(p) := \int_p^\infty D(t)dt$, the surplus of a consumer who pays p.

Consumers do not know the prices charged by individual firms. However, they hold common beliefs about the price distribution. Cumulative distribution function, *F(.)*, gives the consumers' beliefs about the price distribution; the lowest and highest prices on its support are $\underline{p}$ and $\overline{p}$.

Consumers search sequentially with recall, and have a constant marginal search cost. Search is instantaneous. Consumers may observe any number of prices, and pick randomly which firm to sample, from the set of firms whose price they do not know. Denote the *search cost* by σ on $(0,\overline{\sigma})$. The value $\overline{\sigma} < +\infty$, will be defined formally later, and is such that in equilibrium: **(i)** consumers choose to observe at least 1 price, and **(ii)** higher cost firms are constrained by consumers search.

A consumer's *information set* just after his *k*-th search (or return) step, consists of all previously observed prices. A consumer's *strategy* is a stopping rule, s, which says if search should stop or continue, for every search cost, beliefs, and sequence of observations. A consumer's *payoff* is its expected surplus, net of the search expenditure.

2.3 Firms

There is a unit measure continuum of risk neutral firms, which differ only with respect to marginal costs.

Marginal costs are constant and drawn from the twice continuously differentiable cumulative distribution function $\Phi(.)$, with support on $[\underline{c},\overline{c}]$, $0 \le \underline{c} < \overline{c} < D^{-1}(0)$.

[Insert Figure 1 here]

Denote by *p(c)* on $\Re_0^+$, the price of a cost c firm, and denote by $\pi(p(c);c) := (p(c)-c)D(p(c))$, the per consumer profit of a cost c firm. Let $\hat{p}(c) := \arg\max_p \pi(p;c)$. I will refer to $\hat{p}(c)$ as the *monopoly price* of a cost c firm. I assume that $\pi(.)$ is strictly quasi-concave in p. Denote by $\varphi(p(c))$, the expected consumer share of a cost c firm, and denote by $\Pi(p(c);c) := \pi(p(c);c)\varphi(p(c))$, the expected profit of a cost c firm.

Instead of modeling explicitly the firms' entry and exit decisions, I follow an equivalent, but more parsimonious approach, in line with the literature, e.g., Benabou (1993) and MacMinn (1980). I assume that all firms are in the industry. However, firms may price themselves in or out of the market. More specifically, a firm may charge a price higher than the maximum consumers are willing to pay, their *reservation price*, in which case I say that the firm is *inactive*; otherwise the firm is *active*. When indifferent between being active and inactive, a firm chooses the latter.

Consumers know if a firm is active without search.[2] This assumption allows at least two interpretations. The first and more literal interpretation is that consumers have a directory of active firms. The second interpretation is that search costs are mainly informational costs. Reaching firms is easy, eliciting price quotes is harder. Electronic markets are a natural candidate to illustrate both of these cases.

I assume that a cost $\bar{c}$ firm loses money if it charges $\hat{p}(\underline{c})$, i.e., $\hat{p}(\underline{c}) < \bar{c}$ (Figure 1). This allows the possibility that higher cost firms may be inactive in equilibrium.[3] One can interpret this assumption as meaning that cost heterogeneity among firms is large.

A firm's *strategy* is a rule that says which price a firm should charge, for every marginal cost. A firm's *payoff* is its expected profit.

2.4 Equilibrium
The solution concept is a refinement of a symmetric pure strategy Nash equilibrium.

An *equilibrium* is: a stopping rule, consumer beliefs, and a pricing rule, $\{s^*, F^*, p^*\}$, such that:[4]

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[2] Thus, F gives the consumers' beliefs about the prices of active firms. In my approach, assuming that consumers know which firms are active, is equivalent to assuming that consumers know which firms are in the market, when the firms' entry or exit decisions are modeled explicitly. The search literature usually assumes that consumers: (i) know which firms are *active*, i.e., which firms are *in* the industry, (ii) know the price distribution, but (iii) don't know the prices of individual firms. There is a branch of the search literature going back to Rothschild (1974), where consumers learn through search, both individual prices, and the price distribution. This literature addresses different issues from those addressed in my paper.

[3] Case $\hat{p}(\underline{c}) < \bar{c}$, considered by Benabou (1993) and MacMinn (1980), is thus more encompassing than the alternative case $\bar{c} \leq \hat{p}(\underline{c})$, considered by Reinganum (1979).

[4] I follow Bagwell & Ramey (1994). Restrictions (E.1) and (E.2) are variations of Kreps & Wilson's (1982) concepts of *sequential rationality* and *consistent beliefs*, which generalize subgame perfection to incomplete information games, and rule out unreasonable Nash equilibria. The first implies that consumers behave optimally at every information set, given their beliefs about the firms' strategies. The second implies that consumers do not change their beliefs upon observing the price of any single firm, and thus any finite set of firms, and that on the equilibrium path, consumers' beliefs agree with the price distribution

(E.1) Consumers choose s^* to maximize the net expected consumer surplus, given σ, and given the previously observed prices and their conjecture of the price distribution at the unsearched firms, conditional on any observed information, i.e., given F^*;

(E.2) Consumers believe firms choose prices independently and maintain this assumption throughout the search process;

(E.3) Firms choose p^* to maximize expected profit, given s^* and c;

(E.4) Beliefs F^* agree with the price distributions induced by p^* and Φ.

3 Characterization of Equilibrium

In this section I construct the equilibrium by working backwards. The consumers' equilibrium search behavior consists of holding a reservation price. Low cost firms are always active and charge their monopoly price. High cost firms are sometimes active, others inactive; when active, high cost firms charge the minimum of the consumers' reservation price and their monopoly price.

3.1 The Search Game

In this sub-section I characterize the search equilibrium.

The consumers' optimal strategy consists of holding a reservation price, ρ, which equates the expected marginal benefit of search to the search cost:[5]

$$\int_{\underline{p}}^{\rho} \left[S(p) - S(\rho) \right] dF(p) = \sigma \tag{1}$$

That is, the consumers' optimal strategy is to terminate search and buy, if and only if offered a price no higher than ρ.

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induced by the firms pricing strategies.

[5] See Benabou (1993) or Reinganum (1979). Search occurs if either consumers get the first price quote for free, or the search cost is "small enough". Given (E.1) consumers optimize with respect to beliefs, which given (E.2) do not depend on observed prices. Thus, the consumers' search problem can be solved through dynamic programming. Under my assumptions sequential search is optimal (Morgan & Manning (1985), Proposition 3).

Inspection of (1) shows that for any strictly positive search cost, $\sigma > 0$, the reservation price is strictly higher than the lowest price charged in the market, $\underline{p} < \rho$. Costly search, gives firms market power, since it leads consumers to accept prices above the minimum charged in the market.

Value $\bar{\sigma}$ can now be defined by $\int_{\underline{p}}^{\widehat{p}(\overline{c})} [S(p) - S(\widehat{p}(\overline{c}))] dF^*(p) - \bar{\sigma} \equiv 0$. For σ on $(0, \bar{\sigma})$, in equilibrium: **(i)** consumers search, and **(ii)** higher cost firms are constrained by consumer search, $\rho < \widehat{p}(\overline{c})$.[6]

3.2 The Pricing Game

In this sub-section I characterize the equilibrium prices.

Let $\widehat{c}(p) := \widehat{p}^{-1}(p)$.[7] Denote by n, the measure of active firms. If a firm charges a price higher than the reservation price, it makes no sales; if it charges a price no higher than the reservation price it gets an expected consumer share of $1/n$. Thus:

$$\varphi(p) = \begin{cases} 0 & \Leftarrow \rho < p \\ \dfrac{1}{n} & \Leftarrow p \leq \rho \end{cases}$$

The next Lemma characterizes the equilibrium prices.

Lemma 1: Let $\rho < \widehat{p}(\overline{c})$. **(i)** If the reservation price is lower than a firm's marginal cost, the firm is inactive. If the reservation price is no lower than a firm's marginal cost, the firm charges the minimum of the reservation price, and its monopoly price. **(ii)** Given ρ and Φ:

$$F^*(p; \rho) = \begin{cases} \dfrac{\Phi(\widehat{c}(p))}{\Phi(\rho)} & \Leftarrow p < \rho \\ 1 & \Leftarrow \rho \leq p \end{cases}$$

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[6] The analysis of this paper generalizes to all σ on $(0, +\infty)$. For $\bar{\sigma} < \sigma$, equilibria where $\widehat{p}(\overline{c}) < \rho$ may occur. At these equilibria no firm is constrained by consumer search. All types of firms, irrespective of how inefficient they are, charge their monopoly price, $\widehat{p}(c)$. The focus on σ on $(0, \bar{\sigma})$, i.e., on $\rho < \widehat{p}(\overline{c})$, allows one to discard these uninteresting equilibria.

[7] Strict quasi-concavity of $\pi(.)$ implies that $\widehat{p}(c)$ is a singleton. Demand being strictly decreasing implies that $\widehat{p}(.)$ is strictly increasing.

Proof: (i) The proof is in the appendix. Next I provide a sketch. First, I show that $p^*(.)$ is increasing in c. Second, I show that due to costly search, $\sigma > 0$, and (E.2), the lowest cost firm enjoys full market power, $\rho > p^*(\underline{c})=\widehat{p}(\underline{c})$. And third, I show that lower cost firms also enjoy full market power, $p^*(c)=\widehat{p}(c)$ for c on $[\underline{c},\widehat{c}(\rho)]$; higher cost firms are constrained by consumer search, and may be active, $p^*(c) = \rho$ for c on $(\widehat{c}(\rho),\rho]$, or inactive, $p^*(c) > \rho$ for c on $(\rho,\overline{c}]$. **(ii)** For ρ on $[c,\widehat{p}(c)]$, $p^*(c) = \rho$. For $\rho > \widehat{p}(c)$,

$$F^*(p;\rho) = Prob[P \le p | P < \rho] = Prob[\widehat{p}(C) \le p|\widehat{p}(C) < \rho] = Prob[C \le \widehat{c}(p)| C < \rho] = \Phi(\widehat{c}(p))/\Phi(\rho). \qquad §$$

[Insert Figure 2 here]

The firm with the lowest cost, $\underline{c}$, charges the lowest price. Given that consumers accept prices above the minimum charged in the market, $\underline{p} < \rho$, firm $\underline{c}$ is never constrained by consumer search and charges its monopoly price, $\widehat{p}(\underline{c})$ (Figure 2). Firms with a low enough cost, i.e., whose cost belongs to $[\underline{c},\widehat{c}(\rho)]$, also charge their monopoly price, $\widehat{p}(c)$. Firms with a higher cost may also benefit from the market power generated by costly search, by charging a higher price than lowest cost firm. However, they are disciplined by consumer search. If the reservation price is *high*, $\overline{c} < \rho$, firms whose cost belongs to $(\widehat{c}(\rho),\overline{c}]$, charge the reservation price. If the reservation price is *low*, $\rho \le \overline{c}$, firms whose cost belongs to $(\widehat{c}(\rho),\rho]$, charge the reservation price, and firms whose cost belongs to $(\rho,\overline{c}]$, are inactive, i.e., charge a price on $(\rho,+\infty)$. Since this is a 1 period model, a firm being inactive is equivalent to *exiting* the industry. From Lemma 1: $n = \Phi(\rho)$.

The equilibrium cumulative price distribution function, $F^*(.)$ is induced by $\Phi(.)$ and $p^*(.)$. Function $F^*(.)$ has a mass point at $p = \rho$, and as higher cost firms exit the industry, it shifts in the first-order stochastically dominated sense.

3.3 The Equilibrium Reservation Price Correspondence

In this sub-section I develop the equilibrium reservation price correspondence.

Using Lemma 1, (1) can be written as:

$$\frac{1}{\Phi(\rho)} \int_{\hat{p}(\underline{c})}^{\rho} [S(p) - S(\rho)] d\Phi(\hat{c}(p)) = \sigma \tag{2}$$

For (r, σ) on $[p(\underline{c}), p(\overline{c})] \times (0, \overline{\sigma}]$, let $\Psi(r; \sigma) := \int_{\underline{p}}^{r} [S(p) - S(r)] d\Phi(\hat{c}(p))/\Phi(r) - \sigma$.

The next Lemma shows that the reservation price is consistent with the equilibrium price distribution.

Lemma 2: Given F^*, and for every σ on $(0, \overline{\sigma})$, consumers choose ρ on $(\hat{p}(\underline{c}), \hat{p}(\overline{c}))$.

Proof: Since $\Psi(.)$ is continuous with respect to r, and for all σ on $(0, \overline{\sigma})$, $\Psi(\hat{p}(\underline{c}); \sigma) < 0 < \Psi(\hat{p}(\overline{c}); \sigma)$, it follows from the intermediate value theorem that for all σ on $(0, \overline{\sigma})$, there is ρ on $(\hat{p}(\underline{c}), \hat{p}(\overline{c}))$ such that $\Psi(\rho; \sigma) \equiv 0$. §

Denote by $R(.)$, the equilibrium reservation price correspondence, $\rho = R(\sigma)$, defined implicitly by (2).[8]

4 Multiplicity

In this section I discuss multiplicity of equilibria.

From equation (2), if high cost firms are inactive, $\rho \leq \overline{c}$, or in the alternative interpretation if exit occurs, the reservation price affects the measure of active firms, $n = \Phi(\rho)$. The measure of active firms affects the price distribution, $F^*(p; \rho) = \Phi(\hat{c}(p))/\Phi(\rho)$. In equilibrium, the price distribution agrees with the consumers' beliefs about the price distribution, $F(p) = F^*(p; \rho)$. Thereby, the measure of active firms affects the reservation price. The reservation price and the measure of active firms are mutually reinforcing. Given this complementarity, is equilibrium unique? That is, for a given level of the search cost, is there a unique equilibrium level for the reservation price, or are there several equilibrium levels for the reservation price? In the remainder of this section I address this question.

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[8] Mapping $R(.)$ is defined almost everywhere, since given the differentiability of $D(.)$ and $\Phi(.)$, the set of critical values of $\psi(.)$ has Lebesgue measure 0.

I start with a motivating example.

Example 1: Let $D(p) = 998 - p$ and $\Phi(c) = c/\overline{c}$. Thus, $\hat{p}(c) = (998 + c)/2$, $\hat{p}(0) = 499$, and

$$\Psi(\rho;\sigma) = \frac{1}{\Phi(\rho)}\int_{\hat{p}(\underline{c})}^{\rho}\{[S(p) - S(\rho)]\}d\Phi(\hat{c}(p)) - \sigma = \begin{cases} -\sigma + \dfrac{621{,}257{,}495}{3\overline{c}} - \dfrac{996{,}004}{\overline{c}}\rho + \dfrac{1{,}497}{\overline{c}}\rho^2 - \dfrac{2}{3\overline{c}}\rho^3 & \Leftarrow \ \overline{c} < \rho \\[4mm] -\sigma - 996{,}004 + \dfrac{621{,}257{,}495}{3\rho} + 1{,}497\rho - \dfrac{2}{3}\rho^2 & \Leftarrow \ \rho \le \overline{c} \end{cases}$$

Thus,

$$\frac{\partial\Psi(\rho;\sigma)}{\partial p} = \begin{cases} -\dfrac{996{,}004}{\overline{c}} + \dfrac{2{,}994}{\overline{c}}\rho - \dfrac{2}{\overline{c}}\rho^2 & \Leftarrow \ \overline{c} < \rho \\[4mm] 1{,}497 - \dfrac{621{,}257{,}495}{3\rho^2} - \dfrac{4}{3}\rho & \Leftarrow \ \rho \le \overline{c} \end{cases}$$

The upper expression of $\partial\Psi/\partial\rho$ is positive for $\overline{c} < \rho < \hat{p}(\overline{c})$. The lower expression is positive for $\rho < 951$, and negative for $\rho > 951$. Next I consider 2 cases. **(i)** Let $\overline{c} = 950$. Thus, $0 < \partial\Psi/\partial p$ for $\hat{p}(0) < \rho \le \hat{p}(950) = 974$, and $\overline{\sigma} = 43{,}304$. **(ii)** Let $\overline{c} = 986$. Thus, $\hat{p}(986) = 992$, and $\overline{\sigma} = 41{,}987$. Mapping $R(.)$ is sketched in Figure 3. For σ on $(0;41{,}934)$, equilibrium is unique, and for σ on $[41{,}935;41{,}987]$ there are multiple equilibria. For example, $R(41{,}950) = \{916;\ 985;\ 987\}$. §

[Insert Figure 3 here]

The next Proposition provides two sufficient conditions for uniqueness. The first condition is also necessary.

Proposition 1: (i) For all ρ on $(\hat{p}(\underline{c}),\overline{c}]$,

$$D(\rho)\left(\frac{\Phi(\hat{c}(\rho))}{\Phi(\rho)}\right) - \left(\frac{\Phi'(\rho)}{\Phi(\rho)}\right)\int_{\hat{p}(\underline{c})}^{\rho}D(p)\left(\frac{\Phi(\hat{c}(p))}{\Phi(\rho)}\right)dp \ge 0 \tag{3}$$

if and only if, $R(\sigma)$ is a singleton for all σ on $(0,\overline{\sigma})$. **(ii)** There is a σ^m on $(0,\overline{\sigma})$, such that for all σ on $(0,\sigma^m)$, $R(\sigma)$ is a singleton.

Proof: First note that after integrating by parts:

$$\frac{\partial \Psi(r;\sigma)}{\partial r} = \begin{cases} D(r)\Phi(\widehat{c}(r)) & \Leftarrow \quad \overline{c} < r \\ D(r)\dfrac{\Phi(\widehat{c}(r))}{\Phi(r)} - \dfrac{\Phi'(r)}{\Phi(r)}\displaystyle\int_{\widehat{p}(\underline{c})}^{r} D(p)\dfrac{\Phi(\widehat{c}(p))}{\Phi(r)}\,dp & \Leftarrow \quad r \le \overline{c} \end{cases} \tag{4}$$

(i) Next I prove the sufficiency. If (3) holds, from (4) $\partial \Psi(r;\sigma)/\partial r \ge 0$, for all r on $\left(\widehat{p}(\underline{c}),\overline{c}\right]$. Given the

differentiability of $D(.)$ and $\Phi(.)$, the set of values of r for which (3) holds as an equality is countable.

The proof of the necessity is in the appendix. **(ii)** The proof is in the appendix. Next I provide a sketch.

First, I show that since $\partial\Psi(\widehat{p}(\underline{c});\sigma)/\partial r > 0$ for all σ, and given the continuity of $\partial\Psi(.)/\partial r$ with respect to

r, there is a r^m on $\left(\widehat{p}(\underline{c}),\overline{c}\right]$, such that, if $r < r^m$, then $\partial\Psi(r;\sigma)/\partial r > 0$. Second, I show that there is a σ^m

on $\left(0,\overline{\sigma}\right)$, such that $\Psi(r^m(\sigma^m);\sigma) \equiv 0$. Third, I show that for every σ on $\left(0,\sigma^m\right)$, there is a unique ρ on

$\left(\widehat{p}(\underline{c}),r^m\right)$ such that $\Psi(\rho;\sigma) \equiv 0$.   §

[Insert Figure 4 here]

Changes in the reservation price have two opposing effects. First, given the cost distribution of

active firms, an increase in the reservation price reduces the size of the mass point at $p = \rho$ (Figure 4 (a)).

Fewer firms are constrained by consumer search. This first effect increases the marginal benefit of search,

and is present in Reinganum (1979). Second, an increase in the reservation price increases the measure of

active firms, which shifts the cost distribution of active firms in the first-order stochastic dominated sense,

i.e., to the right (Figure 4 (b)). This second effect decreases the marginal benefit of search, and is

responsible for the possibility of multiple equilibria. These two effects correspond to the first and second

terms on the left-hand side of (3). Condition (3) ensures that the first effect dominates the second effect,

and thus that (2) is monotonic with respect to the reservation price.

Uniqueness has a simple characterization. If the equilibrium is unique, then (2) must be monotonic

with respect to the reservation price. And if (2) is monotonic with respect to the reservation price, then

equilibrium must be unique. In other words, uniqueness of the equilibrium is equivalent to monotonicity of

(2) with respect to the reservation price.

Whether or not (3) holds globally, if the search cost is small enough, $\sigma < \sigma^m$, the first effect in (3) dominates the second effect, and equilibrium is unique.

The multiplicity discussed in this paper is driven by two assumptions. The first assumption is that cost heterogeneity among firms is large, $\widehat{p}(\underline{c}) \leq \overline{c}$, and some firms may be inactive. Uniqueness emerges, if cost heterogeneity is small, $\overline{c} < \widehat{p}(\underline{c})$, and all firms are always active, as in Reinganum (1979).

The second assumption that drives multiplicity, is that consumers know which firms are active without search. The second effect on the left-hand side of (3) is present, if consumers are able to distinguish between active and inactive firms, and use this information in forming beliefs about the price distribution. Note that the alternative assumption that consumers do not know if a firm is active without search, raises technical problems. When the reservation price is lower than a firm's marginal cost, the firm's best response is to set *any* price no smaller than the reservation price. In this context, it is unclear how the price distribution of all firms, active and inactive, looks like. In any case, when $\overline{c} < \rho$, the equilibrium cumulative price distribution function, $F^*(.)$, would still have a has a mass point at $p = \rho$.

When there are multiple equilibria, small changes in the search cost may lead to large discontinuous changes in the reservation price. This occurs, e.g., when a decrease in the search cost below σ^m leads to a jump from the upper to the lower branch of the equilibrium selection correspondence (Figure 3).

I focus on $\sigma < \overline{\sigma}$. But multiplicity may still occur if $\overline{\sigma} < \sigma$, i.e., if $\widehat{p}(\overline{c}) < \rho$ (Figure 3).

5 Extensions

In this section I discuss two extensions.

When there are multiple equilibria, typically, comparative statics are indeterminate, in the sense that they depend on the type of equilibria chosen. However, one can use local stability with respect to Cournot's adjustment process (Fudenberg & Tirole (1992)) as an equilibrium selection criterion, and then

do local comparative statics.[9] It is straightforward to show that condition (3) is necessary for local stability. In terms of Example 1 and Figure 3, equilibria $R(41,950)=\{916; 987\}$ are locally stable, and equilibrium $R(41,950)=\{985\}$ is locally unstable.

The results of section 4 generalize to the case of consumer heterogeneity with respect to the search cost (Benabou (1993)). Suppose that consumers differ with respect to search costs, which are drawn from some twice continuously differentiable cumulative distribution function, with support on $[\underline{\sigma}, \overline{\sigma}]$, $0 \leq \underline{\sigma} < \overline{\sigma}$. For each type of consumer, the reservation price is determined by equation (1). Consumers with a higher search cost hold a higher reservation price. The distribution of search costs, maps into a distribution of reservation prices, with support on the interval $[\underline{\rho}, \overline{\rho}]$, where $\underline{\rho}$ and $\overline{\rho}$ are the reservation prices of the lowest and highest search cost consumers, respectively. To extend the results of section 4, apply the analysis of Proposition 1 to the consumer with the highest search cost, $\overline{\sigma}$, i.e., to the consumer with the highest reservation price, $\overline{\rho}$.

6 Welfare Analysis: A Coordination Failure

In this section I discuss welfare analysis, and a potential coordination failure.

I start by ranking the equilibria by welfare, i.e., expected profits plus expected consumer surplus. I focus on locally stable equilibria, i.e., I assume (3) holds locally.

The next Proposition ranks the industry profits and the consumers' surplus.

Proposition 2: Let $\rho \leq \overline{c}$. **(i)** The expected industry profits are decreasing in the reservation price. **(ii)** If (3) holds, the expected consumer surplus is decreasing in the reservation price.

Proof: See the appendix.   §

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[9] Consider the following adjustment process, consisting of a succession of rounds, each composed of two stages. In the first stage, firms choose prices, which are a best response to consumers' reservation price of the previous round. In the second stage, consumers choose a reservation price, which is optimal given the measure of active firms of that round. A steady state of the adjustment process is an equilibrium of the model. An equilibrium is *locally asymptotically stable* for the adjustment process, if there exists a neighborhood of the equilibrium such that for any initial point on that neighborhood, the adjustment process converges to the equilibrium; otherwise an equilibrium is *unstable*.

A lower reservation price increases efficiency by inducing exit, and thereby redistributing production from high to low cost firms. Consider a search cost σ', for which there are 2 different reservation price equilibrium levels ρ' and ρ'', i.e., ρ' and ρ'' belong to $R(\sigma')$, with $\rho' < \rho''$ (Figure 3). Suppose the industry moves from the equilibrium in which the reservation price is high, ρ'', to the equilibrium in which the reservation price is low, ρ'. Firms on $[\widehat{c}(\rho'),\widehat{c}(\rho''))$ lose expected profit, $\pi(\rho;c)/\Phi(\rho)$. However, their loss is more than compensated, first by the increase in the expected consumer surplus, and second by the increase in the expected profits of the firms on $[\underline{c},c(\rho'))$. Regarding the first effect, the consumers which patronized the firms that exit, pay a lower price at ρ'. The deadweight loss decreases. The remaining consumers face a price distribution shifted to the left. Regarding the second effect, for a given price, firms on $[\underline{c},c(\rho'))$ have a higher per consumer profit than firms on $[\widehat{c}(\rho'),\widehat{c}(\rho''))$. As high cost firms exit and their consumer share is redistributed to low cost firms, the lost profits of high cost firms are replaced by the higher profits of low cost firms. As a consequence, the expected industry profit increases.

An equilibrium in which the reservation price is low is *potentially* Pareto superior to one in which the reservation price is high. Firms and consumers are jointly better off at a equilibrium where the reservation price is low and the measure of active firms small. However, there is no mechanism that ensures that the industry will be at such an equilibrium. Thus, my analysis identifies a potential coordination failure in search markets with free mobility. Given Proposition 1: (ii), this coordination failure can be avoided as long as the search cost is small.

7 Conclusion

In his paper I identified an instance of multiplicity of equilibria in search markets with free mobility. In a model related to Benabou (1993), Reinganum (1979), and MacMinn (1980), I showed that if there is free mobility, there may be multiple equilibria.

Multiplicity occurs because the consumers' reservation price and the measure of active firms are mutually reinforcing. I provide a monotonicity property, which is necessary and sufficient for uniqueness. Multiplicity vanishes, as the search cost gets small.

Multiple equilibria can be ranked by welfare. Lower reservation prices induce exit, and thereby redistribute production from high to low cost firms. However, no mechanism ensures that the industry will be at the equilibrium with the lowest reservation price.

Appendix

In the appendix I prove Lemma 1: (i), the necessity in Proposition 1 (i), Proposition 1 (ii), and Proposition 2.

Lemma 1 (i):
I proceed in 6 steps.

(step 1) $c' < c'' \implies p^*(c') \le p^*(c'')$.

Suppose not, i.e., $p^*(c'') < p^*(c')$. By definition: $\Pi(p^*(c'');\rho,c') \le \Pi(p^*(c');\rho,c')$, and $\Pi(p^*(c');\rho,c'') \le \Pi(p^*(c'');\rho,c'')$. Adding gives $(c'' - c')\left[D(p^*(c'))\varphi(p^*(c')) - D(p^*(c''))\varphi(p^*(c''))\right] \ge 0$, which is false, since $\varphi(.)$ is non-increasing, and $D(.)$ is strictly decreasing. Thus $p^*(c') \le p^*(c'')$.

(step 2) $p^*(\underline{c}) < \rho$.

Follows from $\underline{p} < \rho$ for $\sigma > 0$, and step 1.

(step 3) $p^*(\underline{c}) = \hat{p}(\underline{c})$.

Given step 2 and the definition of $\varphi(.)$, from the perspective of firm $\underline{c}$, $\varphi(p(\underline{c}))$ is given. Thus, only $\pi(.)$ matters to determine $p^*(\underline{c})$. Suppose $p^*(\underline{c}) \ne \hat{p}(\underline{c})$. Consider first $p^*(\underline{c}) < \hat{p}(\underline{c})$. There is a $\varepsilon > 0$ small enough such that $p^*(\underline{c}) + \varepsilon < \rho$. Thus, if firm $\underline{c}$ deviates and charges $p^*(\underline{c}) + \varepsilon$, it loses no customers, and by strict quasi-concavity, $\pi(.)$ rises. Thus, $\hat{p}(\underline{c}) \le p^*(\underline{c})$. Now consider, $\hat{p}(\underline{c}) < p^*(\underline{c})$. If firm $\underline{c}$ deviates and charges $p(\underline{c}) = \hat{p}(\underline{c})$, given (E.2), and by definition of $\hat{p}(\underline{c})$ profit rises. Thus, $p^*(\underline{c}) \le \hat{p}(\underline{c})$, and hence, $p^*(\underline{c}) = \hat{p}(\underline{c})$.

(step 4) $p^*(c) = \hat{p}(c)$ for $c \in \left[\underline{c}, \hat{c}(\rho)\right]$.

As in step 3, since $\hat{p}(c) \le \rho$ for $c \le \hat{c}(\rho)$.

(step 5) $p^*(c) = \rho$ for $c \in \left(\hat{c}(\rho), \rho\right]$.

Suppose $p^*(c) < \rho$. If firm c charges ρ, it loses no customers, and $\pi(.)$ rises, as in step 3. Suppose $\rho < p^*(c)$. Firm c has no sales, whereas if $p^*(c) = \rho$ it has a strictly positive profit.

(step 6) $p^*(c) \in \left(\rho, +\infty\right)$ for $c \in \left(\rho, \bar{c}\right]$.

Firm c has zero profits for $p(c) \in \left(\rho, +\infty\right)$; otherwise it has losses. §

Proposition 1:
(i) (necessity). I prove that if $\exists r \in \left(\hat{p}(\underline{c}), \bar{c}\right]$:

$$D(r)\big(\Phi(\hat{c}(r))/\Phi(r)\big) - \big(\Phi'(r)/\Phi(r)\big)\int_{\hat{p}(\underline{c})}^{r} D(p)\big(\Phi(\hat{c}(p))/\Phi(r)\big)dp < 0 \qquad \text{(I)}$$

, then, $\exists \sigma \in (0,\overline{\sigma})$: $R(\sigma)$ is not a singleton, i.e., rather than proving $B \Rightarrow A$, I prove $\sim A \Rightarrow \sim B$. I proceed in 12 steps.

(step 1) $\exists r \in (\hat{p}(\underline{c}),\overline{c}]$, $\exists \varepsilon > 0$: (I) holds for $(r - \varepsilon, r + \varepsilon)$.

Follows from (I), and the continuity of $D(.)$, $\hat{c}(.)$, and $\Phi(.)$.

(step 2) given (I), $\exists r \in (\hat{p}(\underline{c}),\overline{c}]$, $\exists \varepsilon > 0$: $\partial \Psi / \partial r < 0$ for $(r - \varepsilon, r + \varepsilon)$.

Follows from (1).

(step 3) $\exists r_1, r_2 \in (\hat{p}(\underline{c}), \hat{p}(\overline{c}))$, $r_1 < r_2$: $\partial \Psi(r_1,\sigma)/\partial r = \partial \Psi(r_2,\sigma)/\partial r \equiv 0$.

Follows from $\partial \Psi(\hat{p}(\underline{c});\sigma)/\partial r > 0$, $\partial \Psi(\hat{p}(\overline{c});\sigma)/\partial > 0$, step 2, and the continuity of $\partial \Psi / \partial r$ with respect to r. Let r_1, r_2 be largest of such roots.

(step 4) $\Psi(\hat{p}(\underline{c}),0) < \Psi(r_2,0) < \min\{\Psi(r_1,0), \Psi(\hat{p}(\overline{c}),0)\}$.

The first inequality is obvious. The second inequality follows from (I), and the definition of r_1 and r_2. The third inequality follows from the definition of r_2, and $\partial \Psi(\hat{p}(\overline{c});\sigma)/\partial r > 0$.

(step 5) Either:

$$\Psi(\hat{p}(\underline{c}),0) < \Psi(r_2,0) < \Psi(r_1,0) \leq \Psi(\hat{p}(\overline{c}),0) \qquad \text{(II)}$$

, or,

$$\Psi(\hat{p}(\underline{c}),0) < \Psi(r_2,0) < \Psi(\hat{p}(\overline{c}),0) < \Psi(r_1,0) \qquad \text{(III)}$$

(step 6) Given (II), $\exists \sigma_1, \sigma_2 \in (0,\overline{\sigma})$: $\Psi(r_1,\sigma_1) = \Psi(r_2,\sigma_2) = 0$, $\sigma_2 < \sigma_1$.

Follows from (II), the definition of $\overline{\sigma}$, and the intermediate value theorem. Inequality $\sigma_2 < \sigma_1$ follows from the monotonicity of $\Psi(.)$ with respect to σ.

(step 7) Given (II), $\forall \sigma' \in (\sigma_2,\sigma_1)$: $\Psi(r_2,\sigma') < 0 < \Psi(\hat{p}(\overline{c}),\sigma')$, $\Psi(r_2,\sigma') < 0 < \Psi(r_1,\sigma')$.

Follows from the definition of σ_1 and σ_2, and the monotonicity of $\Psi(.)$ with respect to σ.

(step 8) Given (II), $\exists r' \in (r_2, \hat{p}(\overline{c}))$: $\Psi(r',\sigma') \equiv 0$.

Follows from step (7), and the intermediate value theorem.

(step 9) Given (II), $\exists r'' \in (r_1, r_2)$: $\Psi(r'',\sigma') \equiv 0$

As in step 8.

(step 10) Given (II), $\forall \sigma \in (\sigma_2,\sigma_1)$, $\exists r',r''$, $r'' < r'$: $\Psi(r',\sigma) = \Psi(r'',\sigma) = 0$.

Follows from steps 8 and 9.

(step 11) Given (III), $\exists r_3 \in (r_1, r_2)$: $\Psi(r_3,\overline{\sigma}) \equiv 0$

Follows from (III), the definition of $\overline{\sigma}$, and the intermediate value theorem.

(step 12) Given (III), $\forall \sigma \in (\sigma_2,\overline{\sigma})$, $\exists r',r''$, $r'' < r'$: $\Psi(r',\sigma) = \Psi(r'',\sigma) = 0$.

As in steps 8 to 11.

(ii) I proceed in 5 steps.

(step 1) $\exists r^m \in (\hat{p}(\underline{c}),\overline{c}]$: if $r < r^m$, then $\partial \Psi(r;\sigma)/\partial r > 0$, $\forall \sigma \in (0,\overline{\sigma})$.

From $\partial \Psi(\hat{p}(\underline{c});\sigma)/\partial r > 0$, $\forall \sigma \in (0,\overline{\sigma})$, and the continuity of $\partial \Psi(.)/\partial r$ with respect to r, $\exists r' \in (\hat{p}(\underline{c}),\overline{c}]$: if $r < r'$, then $\partial \Psi(r;\sigma)/\partial r > 0$, $\forall \sigma \in (0,\overline{\sigma})$. If $\Psi(r,0) \geq \Psi(r',0)$, $\forall r > r'$, let $r^m = r'$. If $\exists r'' \in (r', \hat{p}(\overline{c}))$: $\Psi(r'',0) < \Psi(r',0)$, then $\exists r''' \in (\hat{p}(\underline{c}),r')$: $\Psi(r''',0) = \Psi(r'',0)$, and let $r^m = r'''$.

(step 2) $\forall \sigma \in (0,\overline{\sigma})$, $\exists \rho \in (\hat{p}(\underline{c}),\hat{p}(\overline{c}))$: $\Psi(\rho;\sigma) \equiv 0$.

Follows from Lemma 2.

(step 3) $\exists \sigma^m \in (0,\overline{\sigma})$: $\Psi(r^m(\sigma^m);\sigma) \equiv 0$.

Follows from step 2.

(step 4) $\Psi(\hat{p}(\underline{c});\sigma) < 0 \leq \Psi(r^m;\sigma)$, $\forall \sigma \in (0,\sigma^m)$.

Follows from step 3 and monotonicity of $\Psi(.)$ with respect to σ.

(step 5) $\forall \sigma \in \left(0, \sigma^m\right)$, $\exists^1 \rho \in \left(\hat{p}(\underline{c}), r^m\right)$: $\Psi(\rho; \sigma) \equiv 0$.

Given step 4 and continuity of $\Psi(.)$ with respect to r, it follows from the intermediate value theorem that $\exists^1 \rho \in \left(\hat{p}(\underline{c}), r^m\right)$. Uniqueness follows from monotonicity of $\Psi(.)$ with respect to r, given step 1. §

Proposition 2:

(i) Integration by parts and $\partial\pi(p^*(c); c)/\partial p = 0$ for $c \in \left[\underline{c}, \hat{c}(\rho\rho\right]$ and $\partial p^*(c)/\partial c = 0$ for $c \in \left(\hat{c}(\rho), \rho\right]$ imply:

$$\partial\left(\int_{\underline{c}}^{\rho}\left[\pi(p^*(c); c)/\Phi(\rho)\right]d\Phi(c)\right)/\partial\rho = \left[\Phi'(\rho)/(\Phi(\rho))^2\right]\int_{\underline{c}}^{\rho}\left[\partial\pi(p^*(c); c)/\partial c\right]\Phi(c)dc < 0$$

(ii) Since $F^*(.)$ has a mass point at $p = \rho$, expected consumer surplus equals:

$$ExC := S(\rho)\left[1 - \Phi(\hat{c}(\rho^-))/\Phi(\rho)\right] + \int_{\hat{p}(\underline{c})}^{\rho}\left[S(\rho)/\Phi(\rho)\right]d\Phi(\hat{c}(\rho))$$

After integrating by parts, and using (3):

$$\partial(ExC)/\partial\rho = -D(\rho) - \left[S(\rho)\left[\Phi(\hat{c}(\rho))/\Phi(\rho)\right] - \left[\Phi'(\rho)/\Phi(\rho)\right]\int_{\hat{p}(\underline{c})}^{\rho}\left[D(p)\Phi(\hat{c}(\rho))/\Phi(\rho)\right]dp\right] < 0 .§$$

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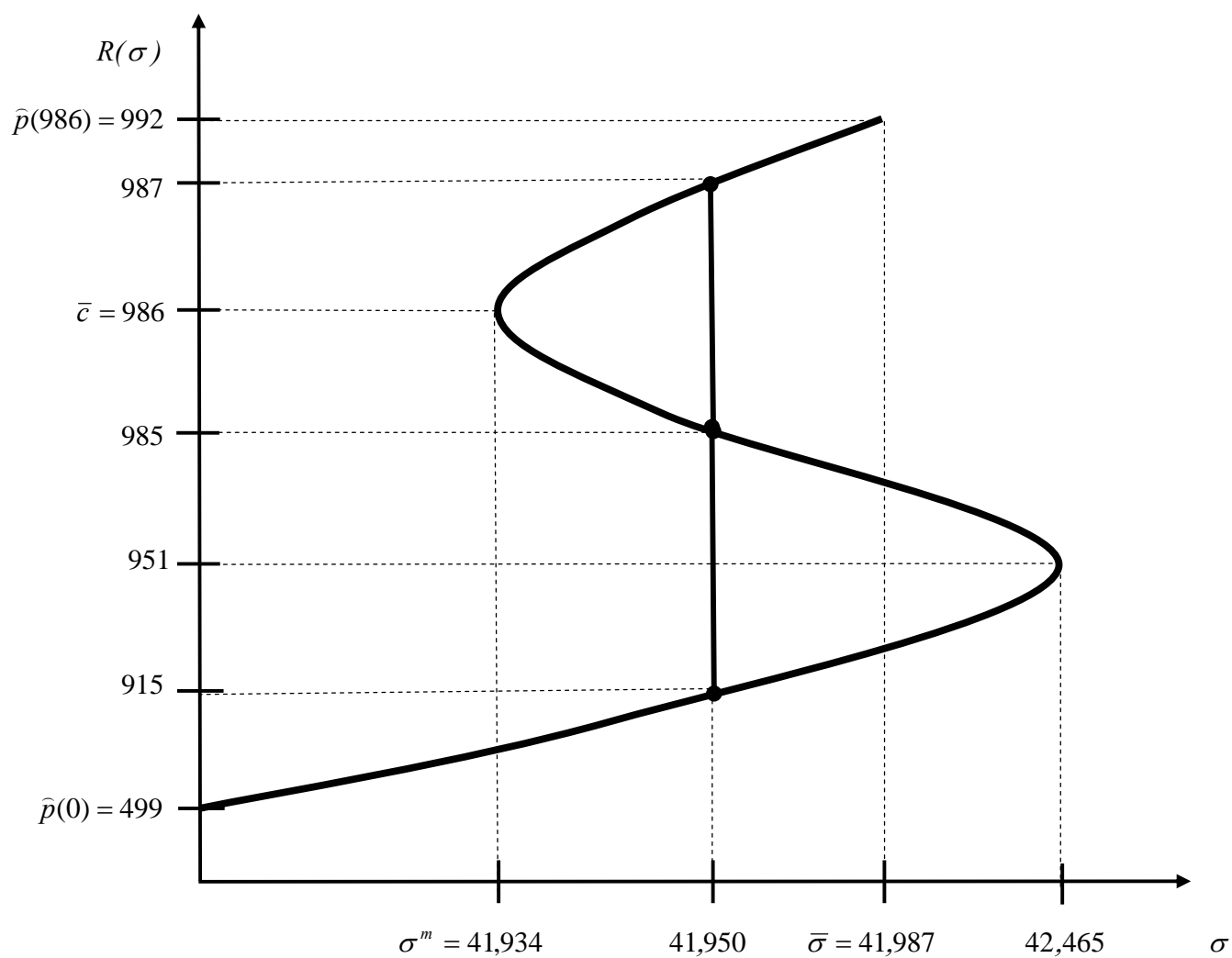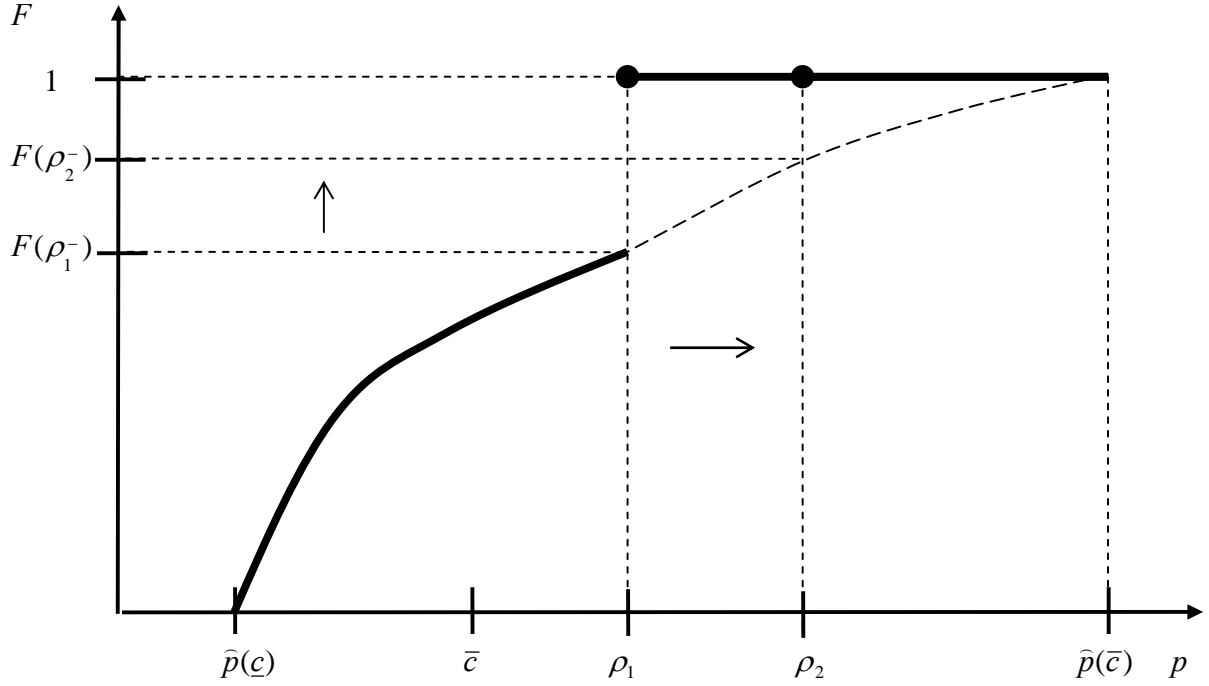
Figure 1: Cost Levels
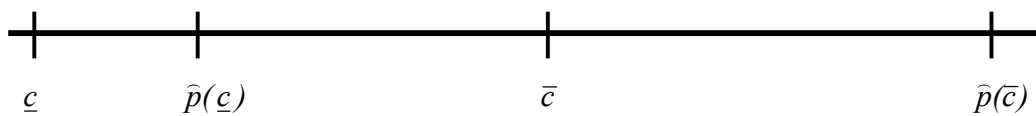
Figure 2: Price Schedule
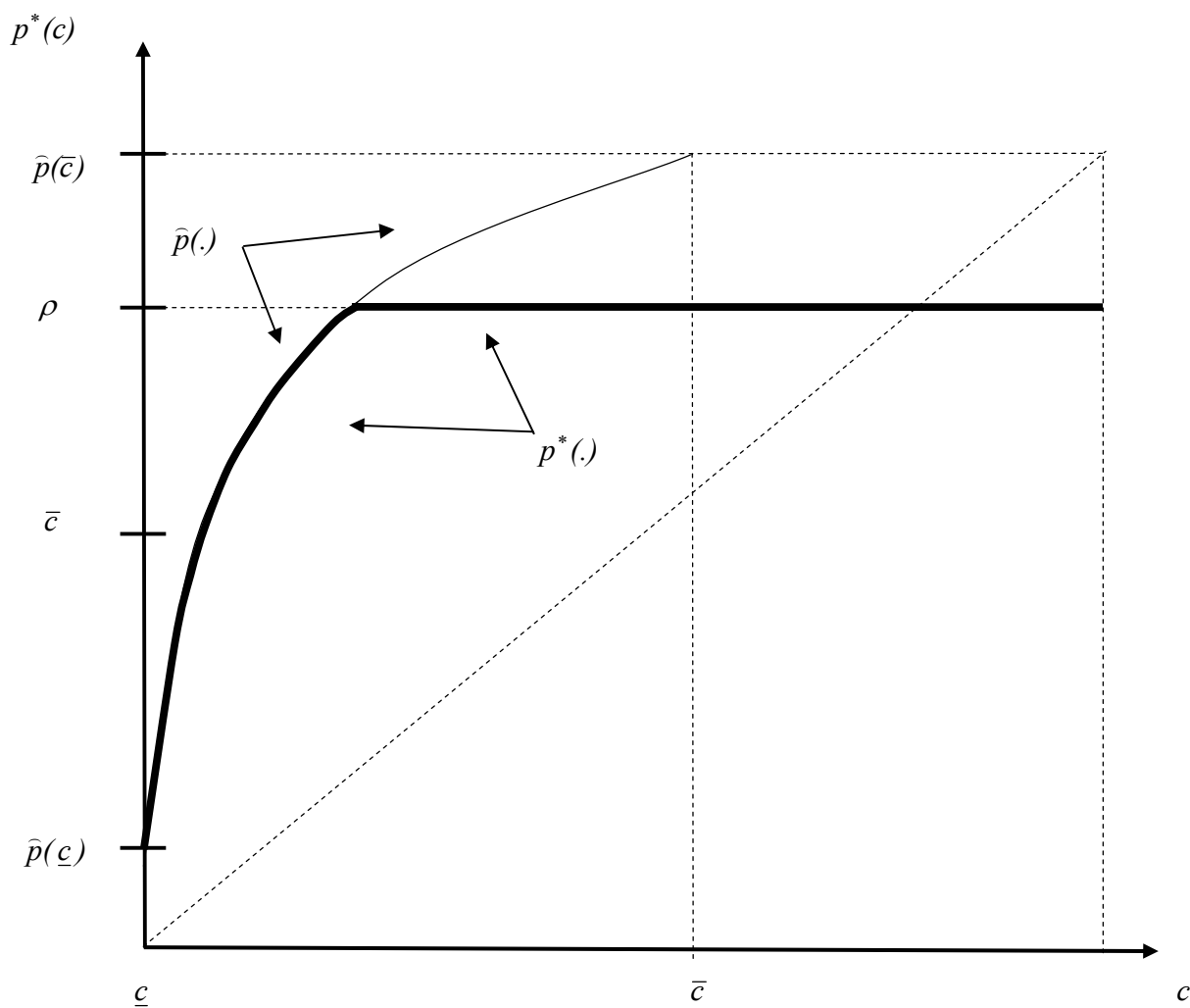
Firms for which $\bar{p}(c) < \rho$ charge $\bar{p}(c)$; firms for which $c < \rho < \bar{p}(c)$ charge ρ; firms for which $\rho < c$ *exit*.
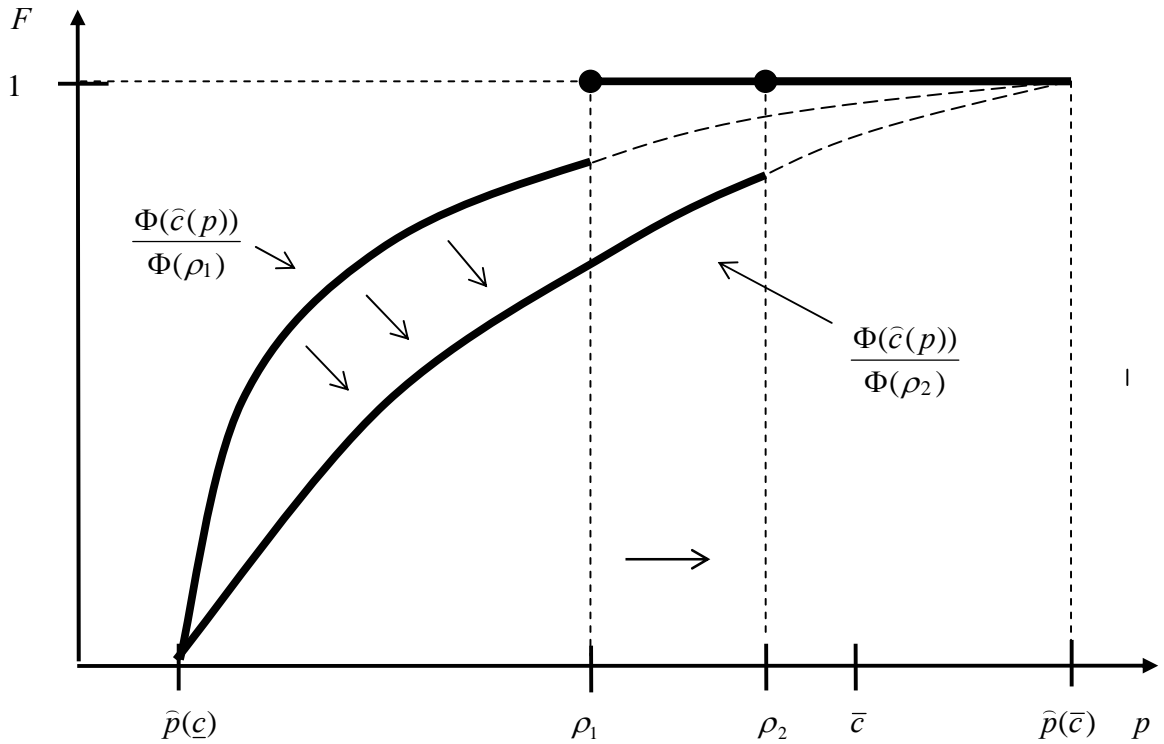
Figure 3: A Coordination Failure

For $\sigma = 41,950$ there are 3 equilibrium reservation prices: 915, 985, and 987. For $\sigma < 41,934$ equilibrium is unique.

Figure 4 (a): An Increse in the Reservation Price, Case $\bar{c} < \rho$

For case $\bar{c} < \rho,$ an increase in the reservation price reduces the mass point of the distribution

Figure 4 (b): An Increse in the Reservation Price, Case $\rho < \bar{c}$

For case $\rho, < \bar{c},$ an increase in the reservation price causes the distribution to shift,

and the impact on the size of the mass point is potentially ambiguous.