Quality Upgrades and the (loss) of Market Power in a Dynamic Monopoly Model

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January 2007
PRELIMINARY- Commment Welcome

[1]We would like to thank the Fuqua Business Associate Fund and Microsoft for their support. Biglaiser also thanks the Portuguese Competition Authority where some of this research was done. This views in this work are solely are own.

1 Introduction

In a classic paper, Coase (1972) conjectured that if a durable goods monopolist could change price very quickly, then the price would fall to marginal cost almost immediately. Thus, the seller has almost no market power if this conjecture is correct. There has been a great deal of work done that examines when the Coase Conjecture holds and how a firm can alter or manage the environment to avoid the loss of market power.

The focus of this literature has been on a firm that sells a single durable good and faces (heterogeneous) consumers who demand only a single unit of the good. While we have learned a great deal, there are many, perhaps to the point of being the norm, durable goods whose qualities are improving over time. One can divide these goods into two classes. One where owning previous qualities of the good impact the ability of a consumer to derive value from newer qualities, and the other when this is not the case, as when the goods are "independent". We refer to the first class as "upgrade goods." Upgrade goods include both items that are commonly thought of as upgrades, and others that are not. Software is the classic example of an upgrade good. But there are many other that can be viewed as upgrade goods. For example, B-52 bombers produced in the 1950s are still in use today and are expected to be in use in 2040, but the plane is quite different than when it first came into use in terms of its electronics, weaponry, and other features.[1] Other defense systems such as tanks and ships are also constantly upgraded. Non-defense goods that are constantly upgraded included airports (terminals and runways), oil refineries, nuclear power plants, cellular network, cable systems, and transportation systems (roads).[2] Goods in the second class of durable goods include items such as television and computer monitors, cellular handsets, and automobiles.

In this paper, we examine an infinite horizon model of upgrade goods where quality is exogenously increasing over time. We find that quality growth in a durable good market can actually lead to a reduction in market power and profits for a monopolist. We employ the simplest possible model needed to demonstrate this result. A monopolist generates a new quality increment in each period and he repeatedly faces the same consumers, all of whom have identical preferences. The

[1] Details can be found on the website http://www.globalsecurity.org/wmd/systems/b-52-life.htm

[2] For details for oil refineries, http://findarticles.com/p/articles/mi_go2264/is_20050,1/ai_n9767023, nuclear power plants, http://www.engineeringtalk.com/news/iic/iic109.html,cellular networks, shttp://whitepapers.techrepublic.com.com/whitepaper.aspx?docid=178659,
and cable systems, http://www.cablesofteng.com/Digital%20Overlay%20Part2.pdf.

monopolist can sell any set of feasible bundles in each period. For a quality increment to be useful to a consumer, he must own all previous quality increments as well -the upgrade payoff structure. posse. We find that in stationary, symmetric, subgame perfect equilibria, the seller's payoff ranges from extracting the entire surplus to receiving only the single period flow value of each quality increment. Thus, we show that even in the case where all consumers are identical, and there is no standard reason for them to earn an information rent, quality growth and buyers who are always in the market can substantially vitiate the market power of the seller. Clearly, this result has policy consequences for many industries including the software industry.

This result is quite surprising relative to the work of Fudenberg, Levine, and Tirole (1985). In that infinite horizon model, they show that a durable goods monopolist who has a good of a single quality will never charge a price below the lowest consumer valuation. Thus, the lowest value consumer is completely extracted. When all consumers are identical, then one would expect that their surplus would be extracted. This logic breaks down as in our model when there is quality growth. Consumers can then obtain positive and even very large shares of the surplus by (implicit) coordination of their behavior. Suppose the consumers are supposed to obtain some positive given level of utility and the monopolist raises the prices from the equilibrium price for the bundle of goods in a period. We can support the equilibrium level of utility by having consumers reject this proposed deviation and receiving a higher utility level in the continuation game. We find a set of supporting utilities that make the monopolist indifferent between giving the buyers the current utility level and delaying to sell till the next period. This gives the buyers the growth in surplus due to the rising quality. After t periods of increasing buyer utility, the buyer utility remains constant and the seller then obtains the surplus growth. The critical t is increasing in the equilibrium utility and the discount factor. We find that our results hold for any discount factor above 1/2.

There is a relatively small on upgrade models, with most of the work involving a finite horizon. Fudenberg and Tirole (1998) examine a two-period model where consumers have different valuations of the good. They focus on how the information structure of the monopolist impacts the pricing strategy for the upgrade product, whether the lower quality is sold in period two, and whether the firm may actually buy back the good it sold in period one. Ellison and Fudenberg (2000) analyze a series of static and two period models. These feature network externalities and a cost to consumers of upgrading the good. They address the issue of whether there is excessive upgrading by the monopolist in the dynamic model and how heterogenous preferences and network externalities interact. We have no direct network externalities in our model and consumers are identical and have no cost of implementing an upgrade. In the finite horizon model version of our model, the monopolist captures all the surplus. Thus, a key feature of our model is that the time horizon is infinite.

There is a large literature on durable goods monopoly. Stokey (1981), Bulow (1982), and Gul, Sonnenschein, and Wilson (1986) prove a version of the Coase Conjecture. In particular, GSW show that if a monopolist's costs are less than the lowest consumer's valuation, then in any stationary equilibrium the conjecture holds. Ausubel and Deneckere (1989) show that if players are sufficiently patient, then any level of average payoff less than the static monopoly payoff can be supported as a subgame perfect equilibrium. Thus, they provide a folk theorem result. Sobel (1991) analyzes a model where consumers only want a single unit of the good, but there is entry of new consumers over time. He proves a folk theorem result for a sufficiently high discount factor. Methodologically, are paper is closest to Sobel, since both feature a market that never closes due to new demand (entry of new consumers and quality growth). Fehr and Kuhn (1995) show that if a monopolist faces a finite set of consumers, then he can completely extract them if he is sufficiently patient, while if there is a smallest unit of account then the Coase Conjecture holds if buyers are sufficiently patient. If both sets are finite, then a folk theorem result holds.

In section 2, we present the model. Benchmarks are generated in section 3 to help differentiate our work from the literature and to understand the implications of the model assumptions. We provide basic results in section 4, where we show that, in equilibrium, whenever a period has a sale consumers always move to the current state of the art and purchase all feasible qualities that they do not possess. In section 5, we examine efficient equilibria in which the monopolist sells the upgrade in the first period that it is available. We show that the monopolist's payoff can range from getting all the surplus to receiving only the single period flow value of each upgrade. In section 6, we show that equilibria can be inefficient in that the sale of upgrades are delayed. For inefficient equilibria, one needs to find approach conditions until there is a sale along with support conditions. We show that there is a critical discount factor, such that the longer the delay, the higher the discount factor must be. We offer conclusions in the final section. All proofs are in the appendix.

2 Model.

We examine an infinite horizon, discrete time model with $t = 1, 2, ...$There is a continuum of identical buyers with a measure of 1 and a single seller. A new perfectly durable good, unit t, becomes available in each period t. All seller costs are 0. Within each period t, the seller can offer prices for any bundle of feasible goods, current quality t and past $1, ...t-1$ qualities. For example, the seller could offer a bundle that includes all feasible qualities $\{1, 2, ...t\}$ at a price p as well as unbundling all qualities by offering quality $\{1\}$ at price p_1, quality $\{2\}$ at a price p_2, and so on. Of course, the seller can withhold some qualities or even make no offer. Thus, any collection of subsets of $\{1, 2, ...t\}$ and associated prices is a feasible offer for the seller. The buyers simultaneously respond to the seller by choosing which bundle(s) to accept in period t. In equilibrium, we will show that a seller need only offer a single set of contiguous qualities, a bundle σ, and a price for that bundle, $p_t(\sigma)$, in period t.

The flow utility that a buyer receives if he possesses units $1, ..., q_t$ in period t is vq_t. We impose the condition that a buyer must have all lower quality units for quality q_t to have value. This is precisely the upgrade structure. Thus, if a buyer holds quality units 1 and 3 but not 2 in a period, then she only receives a flow value v from having the first unit of quality.

Players are all risk neutral and have a common discount factor $\delta < 1$. To avoid several nuisance cases, we assume $\delta \geq 1/2$. Consider an arbitrary sequence of offers, bundles and prices, for each period t. The set of current and previous bundle purchases specifies the current set of quality units held by buyers. For any period t, define q_t as the maximal quality (contiguous units) if a buyer holds units 1,...,q_t but not unit $q_t + 1$. A buyer's payoff is the present discounted value from quality flows net of payments from period t as given by

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t}(vq_\tau - p_\tau)$$

Similarly, the seller's payoff is the present discounted value of revenues from the sales of any sequence of bundles from period t as given by

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t} p_\tau.$$

Note that for any path of qualities and payments, we have the sum of buyers and the seller payoffs as

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t} vq_\tau$$

Thus, the realized joint surplus is determined only by the quality path. Since $q_t \leq t$ for any feasible path, with $q_0 \equiv 0$, the joint surplus is maximized when the maximal quality that buyers have at the end of period t is $q_t = t$. The maximal surplus from date t is then given by

$$\begin{aligned} S_t &= vt + \delta v(t+1) + \delta^2 v(t+2) + ... \\ &= \frac{v(t-1)}{1-\delta} + v\sum_{\tau=1}^{\infty} \tau\delta^{\tau-1} \\ &= \frac{v(t-1)}{1-\delta} + \frac{v}{(1-\delta)^2} \end{aligned}$$

Note that $S_1 = \frac{v}{(1-\delta)^2}$ is the maximal joint surplus at the start of the game. It is the surplus when buyers acquire one new unit in each period, where each new unit has a present discounted value of $\frac{v}{1-\delta}$.

In this paper, we examine subgame perfect equilibria. For our results, it is sufficient to consider equilibria in which (i) buyers follow symmetric strategies and (ii) a stationarity property holds. As is standard, strategies can depend on the history of the game, which is given by the sequence of previous seller offers and buyers' acceptance decisions. If buyers' strategies are symmetric, then any two buyers with the same history must make the same current purchase decision.

In order to define stationarity, we need to introduce the notion of a state. Consider, for instance, the start of the game. This is where the seller has one unit of quality and the buyers have no holdings. We denote this "state" by $(1, 0)$. The seller can offer one unit at some price and, by symmetry, buyers either all accept or all reject the offer. Thus, in period 2, the state is either $(2, 0)$, all buyers rejected the offer at date 1, or $(2, 1)$, all buyers accepted the offer. More generally, define the state (t, Q) by any history that leads to period t where buyers enter the period with maximal quality level Q, (with units 1 through Q).[3]

By symmetry, in response to the seller's offers in state (t, Q), the buyers all move to some higher maximal quality level, Q', or remain at Q. Thus, we can introduce the notion of an upgrade, meaning a bundle of quality units $\{Q + 1, .., Q'\}$ to account for any buyer/seller transaction in period t. Note that any state that a seller can achieve by offering a set of bundles, can also be achieved more simply by offering an upgrade bundle that aggregates the purchases of buyers. By symmetry, only one upgrade bundle is needed. That is, any equilibrium path can be implemented via an upgrade offer structure by the seller: at each state (t, Q), the seller either delays by making no offer or offers one upgrade level $Q' \in (Q + 1, ..., t)$ and an associated price.

[3]By definition a state (t, Q) includes all histories where buyers may also hold any subset of the set $(Q + 2, ..t - 1)$. Whether any non-contiguous quality units are transacted turns out to be unimportant for equilibrium payoffs; see proofs for details. What matters for equlibrium payoffs and paths is when a maximal (contigouous) quality is reached.

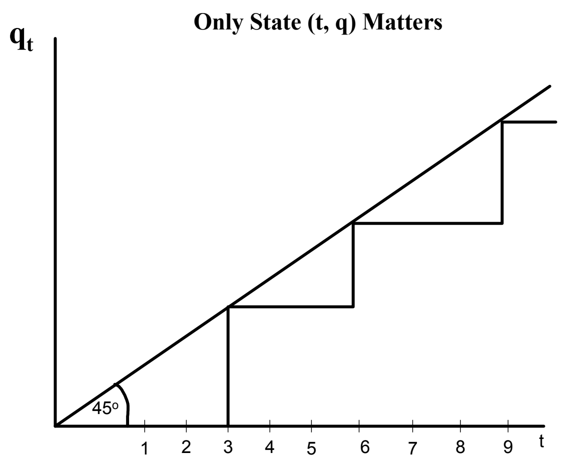

Figure 1:

We define stationarity by the condition that players' strategies depend only on the gap between the maximal feasible quality, t, and the maximal quality that buyers have when they enter a period. That is, stationarity means that if the seller offers σ units at a price p in state $(t, 0)$ for $t = 1, 2, ...$, then he must offer an upgrade from Q to $Q + \sigma$ at the same price p in state (T, Q), provided that the gaps coincide, $T - Q = t$. Furthermore, buyers' accept/reject decisions for a given upgrade are the same in states $(t, 0)$ and (T, Q). This implies that the seller's profits and buyers' utilities satisfy

$$\pi(t, 0) = \pi(T, Q)$$

and

$$u(T, Q) = \frac{vQ}{1 - \delta} + u(t, 0)$$

where $T - Q = t$. Stationarity implies that past prices and paths of qualities that led to state (T, Q), do not matter to players' strategies at state (T, Q).

Figure 1 illustrates the stationary quality path, when three units are sold every third period.

It is worth discussing our definition of stationarity in relation to stationarity assumptions in the durable goods literature. As in GSW (1986) and Sobel (1991), stationarity implies that players' strategies are not affected by the actions of any individual consumer (or set of measure zero consumers). That is, each buyer is negligible in determining next period's state; in this sense individual buyers have no market power. Nodes where buyers are asymmetric, e.g. some buyers have 1 unit of quality and others have none, never can occur along a continuation path since buyers use symmetric strategies. In contrast to the standard durable goods literature, mixing is not needed to support an equilibrium.

The virtue of stationarity is that it simplifies the task of finding an equilibrium by ruling out many forms of history dependence. As we show, the strategic behavior of buyers and sellers in equilibrium necessarily follows a simple cyclical structure when strategies only depend on the quality gap. The risk with stationarity, of course, is that we are ruling out a wide range of equilibrium payoffs. As we show, however, this is not the case in our model: every payoff that can be achieved in equilibrium can be achieved in a stationary equilibrium. Furthermore, the definition of stationarity is flexible enough to allow for both efficient and inefficient equilibria.

Throughout the paper, we use equilibrium to refer to a stationary, symmetric, subgame perfect equilibrium as defined above.

3 Benchmarks for the Quality Growth Model

We begin our analysis by identifying the equilibrium outcomes for several simplified versions of our model. These outcomes provide benchmarks that help illuminate the roles of quality growth, the infinite horizon, and the set of buyers.

3.1 Rental Solution

Consider a market structure in which the seller can only offer one period rental contracts to buyers. In state $(t, 0)$, where 1 through t is the feasible set of qualities (and 0 is the status quo of buyers), there is a unique stationary, subgame perfect, equilibrium. The seller offers t units at a price of $r_t = vt$ and all buyers accept. Buyers are always fully extracted, since the seller can always offer $r_t - \varepsilon$, and it is strictly dominant for buyers to accept for any positive ε. Such an offer provides positive flow surplus to buyers and next period's state does not depend on the current period's outcome. In a rental market, the state is always of the form $(t, 0)$, since buyers can never carry units from one period to the next. Profits $\pi_t = \sum_{\tau=t}^{\infty} \delta^{\tau-1} r_\tau$, coincide with the maximal joint surplus and therefore we have an efficient outcome. The rental market outcome thus reduces to a version of an ultimatum game in a stationary setting. Note that subgame perfection is being employed to rule out non-credible threats in which buyers do not accept positive surplus offers.

3.2 Finite Horizon $T > 1$.

One of the ways that our model differs fundamentally from the earlier work on durable goods is that buyers do not leave the market once they have made a purchase. That is, an infinite horizon implies that buyers will always seek to acquire higher quality units. Let us consider a finite horizon model so that the prospect of acquiring higher quality units is truncated.

One complication with the finite horizon benchmark is that we need to specify how buyers value their quality holdings after the final period. It is helpful to allow for the two extreme cases of (1) units have no value to buyers after period T; (2) each unit has a value of $\frac{v}{1-\delta}$ from period $T+1$, as if the buyer continued to enjoy the surplus flow $v(1+\delta+\delta^2...)$, even though there are no transactions with the seller after period T. Let $z \in \left[0, \frac{v}{1-\delta}\right]$ denote this "scrap value" for each (contiguous) quality unit that a buyer holds after period T.

Consider the final period. Suppose the state is (T, q_{T-1}), where $q_{T-1} \leq T-1$ is the quality held by buyers at the start of period T. Then there exists a unique SGPE outcome in which the seller offers $(T - q_{T-1})$, i.e. an upgrade from q_{T-1} to T units, and prices the upgrade at an extraction level. All buyers will accept the offer. The price for the upgrade is $p_T = (v + z\delta)(T - q_{T-1})$; the flow value of the upgrade is $v(T - q_{T-1})$ and the scrap value from period $T + 1$ is z. Thus, $u_T = (v + z\delta)q_{T-1}$ and the buyers are held to their status quo utility as of the start of period T.

Now consider period $T - 1$ and suppose the state is $(T - 1, q_{T-2})$, where $q_{T-2} \leq T - 2$. Since buyers know that they will not receive any incremental surplus in period T, they will only pay up to $(v + \delta v + z\delta^2)$ for an additional unit of quality in period $T - 1$. The seller clearly prefers to sell the unit in period $T-1$ rather than period T, since waiting sacrifices the flow value of today's consumption. Thus, there exists a unique SGPE outcome in which the seller offers an upgrade of $(T - 1 - q_{T-2})$ units at a price of $p_{T-1} = (v + \delta v + z\delta^2)(T - 1 - q_{T-2})$ and buyers accept. As a result, $u_{T-1} = (v + \delta v + z\delta^2)q_{T-2}$ and the state next period will be $(T, T - 1)$, as the seller moves buyers to the "state of the art" in $T - 1$. Working backwards to period 1, the seller always offers an upgrade to the current state of the art at an extraction price and the equilibrium path reduces to selling each unit of quality when it is first feasible to do so. Note that this outcome does not depend on whether we have a single buyer or a continuum of them. This outcome also prevails if the quality units are independent goods (no upgrade payoff structure). Finally, the finite horizon makes stationarity irrelevant. To summarize, the absence of future transactions implies that the seller captures all of the social surplus.

3.3 Infinite Horizon, Single Buyer

Now, we consider the set of buyers and suppose that we only have a single buyer instead of a continuum. With a single buyer, whenever he makes a purchase the state necessarily changes. We claim that the seller will follow the efficient path, selling the new unit in each period, and price each unit at extraction, $\frac{v}{1-\delta}$. Let us start with a simple example to see why sales occur without delay. Suppose that there is delay and two units are sold in period 2 at price p. By stationarity, this implies

$$\pi_1 = \delta p + \delta^2 \pi_1$$

and

$$u_1 = \delta (2v - p) + \delta^2 \left(\frac{2v}{1-\delta} + u_1 \right)$$

Now, we can apply a modified version of the familiar argument of FLT (1985) to obtain a profitable speed up deviation by the seller. Suppose the seller offered one unit at a price $\hat{p}$ in period 1. If the buyer accepts, then the seller earns $\hat{\pi} = \hat{p} + \delta \pi_1$. The buyer accepts provided that $\hat{u} = v - \hat{p} + \frac{\delta v}{1-\delta} + \delta u_1 > u_1$. Thus, the deviation is profitable for the seller, $\hat{\pi} > \pi_1$, and acceptable to the buyer, $\hat{u} > u_1$, provided

$$\frac{v}{1-\delta} - (1-\delta)u_1 > \hat{p} > (1-\delta)\pi_1,$$

as follows from the above stationarity expressions for u_1 and π_1. Such a $\hat{p}$ exists if and only if

$$\frac{v}{(1-\delta)^2} > u_1 + \pi_1. \tag{1}$$

Note that the left hand side of (1) is S_1, the maximal surplus. Adding the stationarity expressions for u_1 and π_1 and simplifying we have

$$u_1 + \pi_1 = \frac{1}{1-\delta^2} \left[\delta 2v + \delta^2 \frac{2v}{1-\delta} \right] = \frac{2\delta}{1+\delta} S_1$$

which is less than S_1 for $\delta < 1$. Thus, the seller can profitably speed up the candidate equilibrium. Intuitively, the buyer and seller can share the larger surplus of S_1 by selling a unit in period 1 and it is simple to find a mutually beneficial price for that transaction. More generally, we always have $S_t > \delta S_{t+1}$, and the extra surplus allows us to apply a similar speed up argument to any state $(t+1, q)$ with a sale that is preceded by a delay. Thus, starting in any state the continuation path must involve an immediate upgrade to the state of the art. Thus, with a single buyer, the equilibrium path from the start of the game follows the efficient path with a sale every period.

We now argue that this must imply extraction of the buyer. For each state $(t, 0)$ we know that the continuation is an upgrade offer to the state of the art at price p_t for payoffs of $\pi_t = p_t + \delta\pi_1$ and $u_t = \frac{vt}{1-\delta} - p_t + \delta u_1$. Adding, the equation for the joint payoff is

$$\pi_t + u_t = \frac{vt}{1-\delta} + \delta(\pi_1 + u_1)$$

We must have $u_t = \delta u_{t+1}$: if $u_t < \delta u_{t+1}$ the buyer would reject p_t, since the $t+1$ offer is more attractive; if $u_t > \delta u_{t+1}$, then the seller can raise the price and the buyer would still accept. This implies that $u_1 = \delta^{t-1} u_t$. Substituting for u_t in the equation for the joint payoff and simplifying we have

$$\pi_t = \frac{vt}{1-\delta} + \delta(\pi_1 + u_1) - \frac{u_1}{\delta^{t-1}} = \frac{vt}{1-\delta} + \delta\frac{v}{(1-\delta)^2} - \frac{\delta u_1}{\delta^t}.$$

Suppose u_1 is positive. Then as t goes to infinity the required exponential growth in the buyer's utility will eventually push the seller's profit below zero. Obviously this cannot happen in equilibrium. Thus, the buyer is necessarily extracted. The above dynamic linkage of profit and utility over time is an important consequence of surplus growth that we will return to in the analysis of our full model.

The above argument does not extend to a continuum of buyers: an individual buyer cannot change the state, either by delaying or accepting the seller's offer. For example, in state $(t-1, q)$ if a single buyer accepts an offer to move to the state of the art, but no other buyer accepts, then in the next period the state is (t, q). The seller can only earn a profit by making an offer that targets the full mass of buyers with quality q. This strongly contrasts with the single buyer case, where the buyer fully expects the seller to target the offer to his specific quality position.

3.4 Infinite Horizon, No Growth, Continuum of Buyers.

With no growth, the model reduces to the case of a single good. Thus, when all buyers are identical we essentially have a special case of the problem studied by FLT, which allows for buyer valuation heterogeneity. Using simpler versions of the arguments employed above, there is never delay and buyers are always extracted.

These benchmarks demonstrate the robustness of the seller's market power. We now turn our attention to our model, where there is an infinite horizon, growth in quality, and a set of buyers who never leave the market to show how the necessity of extraction breaks down and moreover may lead to almost a complete loss of his market power.

4 Preliminary results

We will provide an explicit equilibrium construction of the buyers and seller's strategies. To streamline the analysis, we will assume that an individual buyer who deviates by not following other buyers in a purchase that increases the maximal buyer quality, will obtain no future additional surplus. Thus, if an individual buyer has the first k units of the good, when all other buyers also have additional contiguous units, then the deviating buyer's continuation payoff is $\frac{vk}{1-\delta}$. There are two interpretations of this continuation payoff. First, the seller ignores individual buyers, measure zero, who differ from the market path. Thus, the missing units necessary for the buyer to benefit from further purchases will never be offered. Alternatively, the seller can always make the necessary units available, thus allowing the individual buyer to achieve parity with other buyers, but price the units at an appropriate upgrade price, so as to extract all the continuation surplus. As we will see from the equilibrium construction, it is also possible to allow for higher buyer continuation values as long as they do not exceed the equilibrium payoff. It will be clear, from the range of payoffs that are supported in equilibrium, that this is an inessential assumption for the equilibrium construction.

The upgrade structure is important for the continuation value of a deviating buyer. Suppose instead of upgrades, the seller is constrained to offer bundles that contain all lower qualities. Then, an individual buyer who lacks previous quality increments always has the option of restoring his position vis a vis other buyers when they make a future purchase. Whenever buyers with a higher status quo quality level are willing to purchase future units, then the deviating buyer will have a strict preference to make such a purchase since he has fewer units. In this setting, a continuation value in excess of current holdings is a necessary property. In any equilibrium without the upgrade structure, buyers cannot be fully extracted. Thus, the upgrade structure does not impose such a direct limit on the seller's market power.

We now provide some basic results that will serve as building blocks for the main analysis. First, we show that by pricing at a very low level relative to v, the seller can induce buyers to make a purchase.

Lemma 1 *(Flow Dominance) Consider any history such that, at the start of period t, all buyers hold the first Q quality units, where $Q \geq 0$, and no buyer holds unit $Q+1$, where $t > Q$. Suppose the seller makes an upgrade offer for units $\{Q+1, ..., t\}$ at price p, where $p < v(t-Q)$. Then, in any continuation, every buyer accepts the upgrade offer.*

The intuition for "flow dominance" is simple. The upgrade from Q to t is priced sufficiently low that that it pays for itself in the current period, since $vt - p > vQ$. Moreover, even if all other buyers were to reject the offer, an individual buyer who accepts is always weakly better off in the future. This follows from (1) the upgrade payoff structure, since an accepting buyer has a flow surplus of at least vt in future periods, and (2) all buyers have the same opportunities for purchasing from the seller, so an accepting buyer always has the option of making the same choices in the future as other buyers. Essentially, a buyer who holds all of the first t units in period $t + 1$ is never at a disadvantage relative to any other buyer.

It then follows directly that the seller must have a positive payoff both at the start of the game and at any point in the future. This is due to quality growth and flow dominance. At any point in time, the seller always has the option of offering a bundle that includes the new quality unit at a (flow dominant) upgrade price.

Lemma 2 *In any equilibrium, the payoff of the seller is at least* $v/(1-\delta)$. *For any history in which all buyers hold quality units* $\{1, ..., Q\}$ *and no buyer holds unit* $Q + 1$ *at the start of period* t, *the continuation payoff of the seller is at least* $v(t - Q) + \delta \frac{v}{1-\delta}$.

It is important to note that the above results are very basic and, as the proofs demonstrate, they do not depend on stationarity or symmetric buyer strategies. This provides a reference point for our equilibrium construction with stationarity and buyer symmetry: we know that, in any equilibrium, the payoff for the seller can never fall below $v/(1-\delta)$. With this reference point in place, the subsequent analysis will always employ stationarity and symmetry.

A simple consequence of a positive seller payoff in any continuation is that the quality gap never grows without bound. That is, all new quality units are eventually sold within some fixed number of periods.

Lemma 3 *In an equilibrium, for any state* (t, Q), *the continuation path has a bounded quality gap.*

Now, we show that stationarity implies that equilibria must have a simple cyclical structure. To see this, we introduce the notion of a t-cycle equilibrium. In a t-cycle equilibrium a sale occurs every t periods, and t units are sold in each sale period. Thus, the states $(1, 0)$ through $(t - 1, 0)$ are delay states with no sales, and state $(t, 0)$ has a sale of units 1 through t. Hence, once a sale occurs in state $(t, 0)$, the gap falls to 1 and the state returns to $(1, 0)$. Note that this includes as a special case the possibility that $t = 1$, where the current quality unit is sold to buyers in every period.

Proposition 4 *Every equilibrium follows a t-cycle equilibrium path: the buyers purchase quality units* $\{1, ..., t\}$ *from the seller in state* $(t, 0)$*, all payments to the seller occur in state* $(t, 0)$*, and the maximal buyer quality is zero until period* t*.*

What makes this argument work is flow dominance and the fact that the seller can profitably deviate by speeding up a cycle that does not have buyers moving to the state of the art in $(t, 0)$. Thus, if the sale to buyers only involves $\tau < t$ units, the seller can feasibly offer these units in state $(t-1, 0)$. By pricing these units at $\widehat{p} = v\tau + \delta p - \varepsilon$, where p is the price for τ units in state $(t, 0)$, a seller improves his payoff if all the buyers accept since

$$\begin{aligned} \hat{p} + \delta\pi(t, \tau) &> \delta\left[p + \delta\pi(t+1, \tau)\right] \Leftrightarrow \\ (v\tau + \delta p - \varepsilon) + \delta^2\pi(t+1, \tau) &> \delta p + \delta^2\pi(t+1, \tau) \Leftrightarrow \\ v\tau &> \varepsilon \end{aligned}$$

where we have substituted for $\hat{p}$ and the fact that (t, τ) is a delay state.

The candidate equilibrium cannot have buyers rejecting this offer. If other buyers reject, an individual will always find it optimal to purchase the deviation offer (for small $\varepsilon > 0$). By accepting, an individual buyer receives $\delta u(t, 0) + \varepsilon$. To see this, note that the deviating buyer does not change the state, so τ units will be offered next period. Since the buyer already has these units, the purchase in period t can be skipped and the buyer will have the same holdings as all other buyers as of $t + 1$. Thus, we have

$$\begin{aligned} v\tau - \delta\hat{p} + \delta v\tau + \delta^2 u(t+1, \tau) &> \delta\left[v\tau - p + \delta u(t+1, \tau)\right] \Leftrightarrow \\ v\tau &> \hat{p} - \delta p = \varepsilon \end{aligned}$$

Thus, her payoff is improved relative to waiting whenever $\varepsilon > 0$. Hence, all buyers rejecting the offer is not an equilibrium continuation. But, as we showed above, when all buyers accept the offer the seller can profit from making the deviation offer. Thus, an equilibrium with sales of τ less than t cannot be supported, since either the seller can profitably speed up or buyers are required to reject a dominating offer.

By contrast, the speed up argument does not apply to a $t-$cycle equilibrium when $t > 1$ for two reasons. The first is feasibility. The seller does not have t units to sell in period $t-1$. Second, an individual buyer who accepts the deviation offer in $t-1$ is not in an analogous position. By acquiring $t-1$ units when no other buyers accept, an individual buyer can no longer safely skip all purchases in state $(t,0)$, since other buyers will be acquiring units 1 through t. For example, if the seller only offers the bundle of units 1 through t, then the deviating buyer will either have to buy the same bundle as the other buyers and pay for the $t-1$ units that were previously purchased or remain at a utility level of $\frac{v(t-1)}{1-\delta}$. As we will show later, this may make it much less profitable for a seller to induce a speed up.

To summarize, a seller must either sell units as soon as they are feasible, thus following the efficient path, or delay to a maximal set of units periodically, inducing an inefficient path. We study the efficient path next, and then the inefficient path in a subsequent section.

The $t-$cycle equilibria and stationarity allow us to introduce the following simplified notation. Because prices and hence profits depend only the gap between maximal feasible quality available and the buyers' quality position, we can define $\pi(T,Q) = \pi(T-Q,0) \equiv \pi_{T-Q}$ for the seller and $u(T,Q) = \frac{vQ}{1-\delta} + u(T-Q,0)$, with $u(T-Q,0) \equiv u_{T-Q}$, for the buyers.

5 (1,0) Efficient Equilibria

In an efficient equilibrium, a good is sold in each period when it first becomes available. In a stationary equilibrium, this occurs at price p_1 in each period. Thus, the firm's profits and consumers' utilities are $\pi_1 = \frac{p_1}{1-\delta}$ and $u_1 = \frac{1}{1-\delta}\left[\frac{v}{1-\delta} - p_1\right]$, respectively. In an efficient equilibrium, the firm and the consumers divide the maximal social surplus: $S_1 = \frac{v}{(1-\delta)^2} = \pi_1 + u_1$.

To derive the equilibrium payoffs, we must make sure that players cannot do better by deviating.[4] To know that a deviation cannot be profitable, we must specify the continuation payoffs from state $(2,0)$ and other "off-equilibrium path" states. By stationarity and Proposition 4, every continuation state's payoff can be determined once we specify the continuation payoffs in all states of the form $(\tau,0)$. We construct continuation payoffs so that in all states $(\tau,0)$, the seller offers τ units at a price p_τ and this is accepted by all buyers. Thus, the next state is

[4] We apply the one-stage-deviation principle to find the set of subgame perfect equilbrium; our model conforms to the necessary requirement of "continuity at infinity," since the limit of $t\delta^t$ is 0 as t goes to infinity (see Fudengberg and Tirole (1991) pp. 108-110).

$(\tau+1,\tau)$, which returns the quality gap to 1; thus, formally the players are back on the equilibrium path of $(1,0)$. The payoffs with a cash-in support at $(\tau,0)$ are $\pi_\tau = p_\tau + \delta\pi_1$ for the seller and $u_\tau = v\tau - p_\tau + \delta u(\tau+1,\tau) = \frac{v\tau}{1-\delta} - p_\tau + \delta u_1$ for the buyers. Note, that from $(\tau,0)$ this is the efficient path and therefore we have $S_\tau = \frac{v_\tau}{1-\delta} + \delta S_1 = \pi_\tau + u_\tau$.

For a continuation equilibrium to follow this cash-in support, we must specify the accompanying buyer and seller strategies. Buyer strategies follow a simple cut-off rule: a buyer accepts the seller offer of price p for σ units in state $(\tau,0)$ if and only if $p \leq p(\sigma,\tau)$. Similarly, it must be optimal for the seller to offer τ at a price p_τ. Thus, we must find both the "cash-in" price p_τ for all $\tau \geq 2$ and cut-off rules $p(\sigma,\tau)$ for all $\sigma \leq \tau - 1$ and all $\tau \geq 2$.

We can support high buyer surplus in the efficient $(1,0)$ path, even though subsequent outcomes also involves an immediate sale. This may be surprising, since the seller knows that buyers will purchase the state of the art next period, and due to discounting would seem to have a profitable speed up opportunity in addition to the added flow value of a purchase today which he could extract. As we will see, when total surplus is growing over time, this logic is not correct, which is quite different than (FLT). That is, we don't need the threat to destroy surplus, an inefficient outcome, to generate high payoffs for buyers. An inefficient support by a delay would eventually break down, since the seller would have an incentive to cash-in once the continuation surplus S_t is large enough. Thus, we use efficient supporting outcomes.

First, we derive the buyer cut-off strategies. Each buyer must accept any offer $p \leq p(\sigma,\tau)$, given that all other buyers are accepting the offer (symmetric strategies). When all other buyers accept the offer, an individual buyer earns $v\sigma - p + \delta u(\tau+1,\sigma)$ by accepting, where as rejecting yields 0 by assumption. Thus, it is an equilibrium for all buyers to accept p for σ units in state $(\tau,0)$, if $v\sigma - p + \delta u(\tau+1,0) \geq 0$ or equivalently $\frac{v\sigma}{1-\delta} + \delta u(\tau+1-\sigma,0) \geq p$. Analogously, it must be the case that an offer of $p > p(\sigma,\tau)$ is rejected by all buyers. Rejecting the offer when all other buyers reject it, yields a payoff of $\delta u(\tau+1,0)$. Accepting an offer when all other buyers reject yields a flow $v\sigma - p$ today plus the option of purchasing the "continuation offer" of $\tau+1$ next period. Thus, an individual buyer optimally rejects if $\delta u(\tau+1,0) > v\sigma - p + \delta \max\left\{\frac{v\sigma}{1-\delta}, u(\tau+1,0)\right\}$. It is convenient to define $g(\sigma,u) \equiv v\sigma + \delta \max\left\{\frac{v\sigma}{1-\delta}, u\right\} - \delta u$ as the "net surplus" value of the option for a buyer if he makes a purchase when the other buyers do not. When the other buyers purchase in period $\tau+1$, the buyer has two options. If $u > \frac{v\sigma}{1-\delta}$, he will make the purchase when the other buyers do, and thus is willing to pay at most the flow value of the units, $v\sigma$. Otherwise, he will not make the purchase and thus be willing to pay up to $\frac{v\sigma}{1-\delta} - \delta u$. Thus, the buyers' cut-off strategy must satisfy

$$g(\sigma, u(\tau+1,0)) \leq p(\sigma,\tau) \leq \frac{v\sigma}{1-\delta} + \delta u(\tau+1-\sigma,0) \tag{2}$$

for all $\sigma < \tau$ and all $\tau \geq 2$. Since $g(\sigma, u)$ is less than or equal to $\frac{v\sigma}{1-\delta}$, the buyers' cut-off strategies always exist. The right hand side of (2) says that prices must be low enough so that rejection is not optimal for an individual buyer, while the lower bound, which is related to flow dominance, says that it is always optimal to accept offers below this level. Note that $g(\sigma, u(\tau+1,0))$ is at least as large as $v\sigma$; flow dominance says that a buyer is always willing to pay at least $v\sigma$.

Given these buyer responses, the seller must find it optimal to offer τ units at price p_τ in state $(\tau, 0)$. The seller can deviate by delaying, $\sigma = 0$, by offering a partial cash-in with $\sigma \in \{1, ..., \tau-1\}$, or by offering τ units at a price different from p_τ. Beginning with partial cash-ins, note that $p(\sigma,\tau)$ is the optimal price choice and it generates a payoff of $p(\sigma,\tau) + \delta\pi(\tau+1,\sigma)$. This implies that for an equilibrium

$$\pi_\tau - \delta\pi_{\tau+1-\sigma} \geq p(\sigma,\tau) \tag{3}$$

for $\sigma = 1, ...\tau - 1$.

The other two deviations are delay and selling τ units at a price different than p_τ. Delay is not optimal, $\sigma = 0$, if $\pi_\tau \geq \delta\pi_{\tau+1}$. Defining $p(0,\tau) \equiv 0$, (3) applies. Finally, buyers must reject a price above p_τ for the seller offering τ units. Letting $p(\tau,\tau) \equiv p_\tau$ this reduces to (3) holding.

Now we are ready to combine the buyer and seller support conditions, expressions (2) and (3), and identify when there exists supporting prices $p(\sigma,\tau)$, such that the cash-in support constitutes a continuation equilibrium. Combining the seller profit expression (3) with the buyer lower bound on prices, the following condition must be satisfied:

$$\pi_\tau - \delta\pi_{\tau+1-\sigma} \geq p(\sigma,\tau) \geq g(\sigma, u_{\tau+1}).$$

Recalling that $S_\tau = \pi_\tau + u_\tau$, we can find supporting prices provided that

$$S_\tau - \delta S_{\tau+1-\sigma} \geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1}). \tag{4}$$

Note that the surplus difference on the left hand side is an exogenous sequence that is increasing in τ. So, as τ grows larger, more units are "on the table" and a larger set of payoff utilities can be supported. Thus, an equilibrium exists if the following lemma holds:

Lemma 5 *There exist supporting prices $p(\sigma,\tau)$ for $\sigma = 1, ...\tau$ for all $\tau > 1$ if the sequence of buyer utilities u_τ for all $\tau \geq 1$ satisfies (4) for all $\sigma = 0, ...\tau$ and all $\tau > 1$.*

The support must hold for all states, which the seller can control by the choices of bundles that he offers. In particular, it must hold for delay states, where $\sigma = 0$, for cash-in states, where $\sigma = \tau$, and for partial cash-in states, where $\sigma \in (1, \tau - 1)$.

We will show that any buyer utility level $u_1 \in [0, \delta S_1]$ can be supported as an equilibrium payoff for any $\delta \geq 1/2$. That is, the seller may be limited to only the flow payoff of v per period which has a present discounted value of $\frac{v}{1-\delta}$. Thus, the seller may only receive the minimum possible payoff (flow dominance). For each u_1 payoff, we will construct an associated supporting path of $u_2, u_3...$ such that the seller will find it optimal to make an acceptable offer to achieve a cash-in outcome in every state. To gain some intuition for how to support this set of u_1 payoffs, we first look at two special cases of support level utilities. First, we assume that the buyers' support utilities are constant, e.g. $u_1 = u_2 = ... = u$. This means that the seller gets all the gains from the surplus growing. What we will show is that this gives the seller an incentive to delay whenever the buyers' utility exceeds $\frac{v}{1-\delta}$, which is less than δS_1 for $\delta > 1/2$. To see this, use the support condition (4) at $\tau = 1$ and $\sigma = 0$, that is a delay in period 1. Simplifying (4), this becomes

$$S_1 - \delta S_2 \geq (1 - \delta)u$$

or

$$\frac{v}{1 - \delta} \geq u.$$

The above inequality is violated if $\frac{v}{1-\delta} < u$. Since the seller is the residual claimant of surplus, the loss from delay is just v. On the other hand, the gain from delay is the saving in utility given to buyers of $(1 - \delta)u$. If $u > \frac{v}{1-\delta}$, the seller prefers to delay and earn $\delta (S_2 - u)$ rather than $S_1 - u$ from selling today. The discounted share of a larger residual will always dominate once u becomes large enough. This is a direct consequence of the growth in surplus due to quality improving.

What this example shows is that the buyers' utilities must be increasing to support higher levels of utilities. Suppose that the buyers' utilities are always increasing, such that the seller is always indifferent between delay and cashing-in, i.e. the support condition holds with equality at $\sigma = 0$ for all τ. Using the support conditions (4), we ask how fast can utilities grow. Let $\sigma = 0$, which implies $g(\sigma, u_{\tau+1}) = 0$ and (4) becomes

$$u_{\tau+1} \geq \frac{u_\tau - S_\tau + \delta S_{\tau+1}}{\delta}$$

or the seller is indifferent between cashing-in in period t and delay in $t + 1$ if

$$u_{\tau+1} = \frac{u_\tau - v\tau}{\delta}. \tag{5}$$

The support condition for $\sigma = t$ is

$$S_t - \delta S_1 \geq u_t - \delta u_1 + g(t, u_{t+1}).$$

For t sufficiently large, it is straightforward to show that $g(t, u_{t+1}) = \frac{vt}{1-\delta} - \delta u_{t+1}$, since $1 > \delta + \delta^t$. That is, $\frac{vt}{1-\delta} > \delta u_{t+1}$ for t sufficiently large. Then the support condition requires that

$$\delta u_1 \geq vt$$

which is violated for t sufficiently large.

What is happening here is the following. The utility growth for the buyers is $u_t - \delta u_{t+1} = vt$. On the other hand, the available surplus in $t+1$ versus t, is $S_{t+1} - S_t = \frac{v}{1-\delta}$. For t sufficiently large, the present value of the buyers' utility must fall if there is no agreement in period t to make the seller indifferent between selling in period t and delaying until period $t+1$. This allows the seller to raise his price.

Thus, we need a support that is increasing, but cannot be increasing either too fast or too long to generate the set of utilities $u_1 \in [0, \delta S_1]$ for any $\delta \geq 1/2$. Now, we generate a sequence of support utilities for buyers and a set of cut-off utilities. The support combines aspects of the two special cases of supports that we just examined. From state $(1, 0)$ to $(T, 0)$, the support will make the seller indifferent between cashing-in in a period and delaying until the following period, while in bigger states the buyer's utility will be constant at u_T. Thus, this is a combination of the two example supports that we just discussed. The support utility sequence is defined by

$$u_\tau = v\tau + \delta u_{\tau+1} \text{ for } \tau = 1, ...T-1 \tag{6}$$

and

$$u_T = u_{T+1}...$$

Clearly, $u_{\tau+1}$ is increasing in u_τ and all u_τ are increasing in u_1. In particular, for a given u_1, then the sequence $(u_2, ..u_T)$ is determined. We define a sequence of utilities that satisfy (6) and where $u_\tau = u_T$ for $\tau > T$ as a *T-stage support.* In particular, the higher u_1 or the larger δ, the longer the period of time that the support utilities must be strictly increasing to implement the higher payoff.

A direct consequence of a *T-stage support* is that we only to need to satisfy the support constraints, equations (4), over the range $\tau = 1, ...T$. This is because, when (4) holds at $\tau = T$, then it necessarily holds at all larger τ whenever the buyer utility remains constant at any level $\overline{u}$. Formally, we have

Lemma 6 *Suppose $u_\tau = \overline{u}$ for $\tau \geq T$. If the support condition (4) holds at T for $\sigma = 0, ...T$, then (4) holds for $\sigma = 0, ...\tau$ for each $\tau > T$.*

Thus, an advantage of a *T-stage support* is that we only have to check a finite set of conditions. That is, the nature of the support when the buyer's utilities are a constant is relatively straightforward to satisfy. We generate two associated lemmas to show the result. The first one allows us to show that if the support works in period τ for all sales σ that are large enough to put the state back in the range where the value of buyer utility is rising, then the support works the next period $\tau + 1$ for any sales σ' that also induce a state where the value of buyer utility is rising. The next lemma demonstrates that if there are only enough sales to buyers such that the continuation state has a constant buyer utility, then the support holds if it holds at $\sigma = 0$ (delay). We will return to the intuition for this lemma when we consider why the seller prefers to cash-in in period T rather than delay.

Using our $T - Stage\ Support$ we have $\pi_\tau = \delta\pi_{\tau+1}$ for all $\tau < T$. This follows directly from simple algebra. As with the second support example (rising utility) , for periods up to T, the buyers are getting all the efficiency gains from the early cash-in, $v\tau$. This can be seen by noting that the efficiency gain is precisely $S_\tau - \delta S_{\tau+1} = v\tau$. Since the difference between u_τ and $\delta u_{\tau+1}$ is exactly $v\tau$, the difference between π_τ and $\delta\pi_{\tau+1}$ must be 0. The cash-in outcome always divides the surplus of S_τ between the buyers and the seller. The payoff to the seller gets larger over time, but this is exactly offset by the discount factor δ.

For $\tau \geq T$, we have $\pi_\tau > \delta\pi_{\tau+1}$, provided that $u_1 \leq (1 - \delta^T)S_1$. It is instructive to understand why the qualification is necessary for $\pi_\tau > \delta\pi_{\tau+1}$ to be satisfied. For periods T and after we ask the question: when does the seller want to cash-in immediately as opposed to waiting to make a sale? The buyers' payoff is now fixed at u_T and never changes. So, the seller can give the buyers u_T now or wait and give them that same payoff next period. Thus, the cost of selling today instead of next period is $(1 - \delta)u_T$. The benefit of selling today is the efficiency gain $v\tau = S_\tau - \delta S_{\tau+1}$. So, cashing now is more profitable than waiting to cash-in tomorrow when

$$v\tau > (1 - \delta)u_T. \tag{7}$$

Otherwise, waiting is better.

Observe that once the seller prefers cashing in now to waiting, he will always prefer to cash-in immediately in any future period. So, $vT > (1 - \delta)u_T$ is necessary and sufficient for the seller to have a strict preference for cashing in now from period T onwards.

To see when the support conditions (4) holds, we use the properties of the $T-Stage\ Support$. Clearly from (6) there is a direct relationship between u_1 and u_T. By simple algebra, $u_1 = (1-\delta^T)S_1$ implies $u_T = \frac{vT}{1-\delta}$. As (7) demonstrates, a higher level of buyer utility cannot be supported by a $T-Stage\ Support$, since the seller would strictly prefer to delay a sale all the way until state $(T+1, 0)$ instead of selling in state $(1, 0)$. For $u_1 < (1-\delta^T)S_1$, the $T-stage$ support always has the property that the seller strictly prefers to cash-in once period T arrives. We will show with $T = 1$, we can support an outcome with buyer payoffs in the interval $\left[0, \frac{v}{1-\delta}\right]$. With $T = 2$, the interval $\left[\frac{v}{1-\delta}, \frac{v(1+\delta)}{1-\delta}\right]$ is supported. In general, at T we support the interval $\left[\frac{v(1+...\delta^{T-2})}{1-\delta}, \frac{v(1+...\delta^{T-1})}{1-\delta}\right]$, which is equivalent to $[(1-\delta^{T-1})S_1, (1-\delta^T)S_1]$.

The outline of the proof to show that we can support every $u_1 \in [0, \delta S_1]$ is as follows

- Pick a utility level u_1 between 0 and δS_1.
- If $u_1 \leq \frac{v}{1-\delta}$, then set $u_\tau = u_1$ for all $\tau > 1$.
- If $u_1 \in \left[\frac{v}{1-\delta}, \delta S_1\right]$, set $u_2 = \left(\frac{u_1 - v}{\delta}\right)$
- If $u_1 \in \left[\frac{v}{1-\delta}, \frac{v(1-\delta^2)}{(1-\delta)^2}\right]$ set $u_\tau = u_2$ $\tau > 2$. If u_2 is larger, then set $u_3 = \left(\frac{u_2 - v}{\delta}\right)$
- Keep following the logic until T where $\frac{(1-\delta^T)}{(1-\delta)^2} \leq u_1 \leq \delta S_1 \leq \frac{(1-\delta^{T+1})}{(1-\delta)^2}$

For now, and future reference, Figure 2 illustrates the relationship between u_1, δ, and T.

It will be useful to define a particular sequence of cut-off utilities. Let $\underline{u}_1 \equiv \left(1-\delta^{T-1}\right) S_1$ and let the other utilities follow (6). For a given T, denote the sequence is defined by $\underline{u}^T \equiv (\underline{u}_1, ...\underline{u}_T)$. We then have the following relationships

Lemma 7 *The T-stage support $\underline{u}^T$ satisfies (i) $\underline{u}_\tau = \left(1-\delta^{T-\tau}\right) S_1 + \frac{v(\tau-1)}{1-\delta}$, (ii) $\underline{u}_\tau \geq \frac{v\tau}{1-\delta}$ if and only if $\delta \geq \delta^{T-\tau}$ and (iii) the $(T+1)$-stage support, $\underline{u}^{T+1}$, satisfies $\underline{u}_{T-1} \equiv \frac{vT}{1-\delta} - v$ and $\underline{u}_T \equiv \frac{vT}{1-\delta}$.*

The discount factor δ determines how large T must be in order to cover the entire range of buyer payoffs $[0, \delta S_1]$. To see this, we first define a set of critical δ cutoffs. Let δ_τ be the root of $\frac{\delta^\tau}{1-\delta} = 1$ for $\delta \in (0, 1)$. The roots have many properties as describe in the following lemma

Lemma 8 *The cut-off sequence δ_τ satisfies (i) $\delta_1 = 1/2$, (ii) $\delta_\tau < \delta_{\tau+1}$, (iii) $\lim_{\tau\to\infty} \delta_\tau = 1$, (iv) If $\delta \in (\delta_{\tau-1}, \delta_\tau)$, then $\delta + \delta^{\tau-1} < 1 < \delta + \delta^\tau$.*

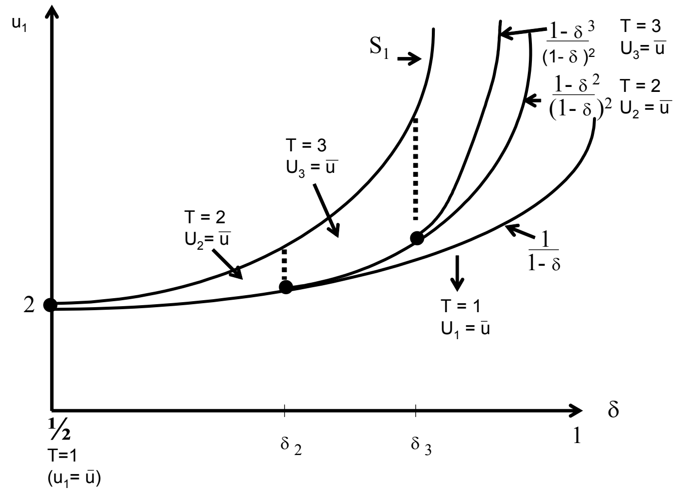

Figure 2:

We use the $T-$stage support in two cases: a) when $\delta \in (\delta_{T-1}, \delta_T]$ and b) when $\delta > \delta_T$. When δ is in the first range, then the largest buyer payoffs can be supported. If δ is in the larger range, then smaller buyer payoffs can be supported with a $T - Stage$ support. The larger the level of δ, the larger T must be to achieve the maximal buyer utility level. We deal with smaller δ by implicitly using a smaller T in the induction proof.

We have

$$\left(1-\delta^T\right) S_1 \lesseqgtr \delta S_1 \Leftrightarrow 1 \lesseqgtr \delta + \delta^T$$

So, applying our two cases, we see that $\delta > \delta_{T-1}$ implies that $\delta + \delta^{T-1} > 1$ and we then have $\left(1-\delta^{T-1}\right) S_1 < \delta S_1$. So, our feasible set of u_1 choices in stage T is always $u_1 \in \left[\left(1-\delta^{T-1}\right) S_1, \delta S_1\right]$.

In case (a), we have $\delta_{T-1} < \delta < \delta_T \Rightarrow \underline{u}_1 < u_1 < \delta S_1 < \left(1-\delta^T\right) S_1$. So , by Lemma 7, we have $u_\tau > \frac{v\tau}{1-\delta}$ for all $\tau = 1, ...T-2$ and $u_{T-1} = u_T = \frac{v(T-1)}{1-\delta}$. As we increase u_1 from $\underline{u}_1$ all the other utilities u_τ grow and at $u_1 = \delta S_1$ we have $\frac{v(T-1)}{1-\delta} < u_T < \frac{vT}{1-\delta}$, since $\delta S_1 < \left(1-\delta^T\right) S_1$ when $\delta < \delta_T$. So, the entire set of $u_1 \in [\underline{u}_1, \delta S_1]$ are covered.

In case (b), we have $\delta > \delta_T \Rightarrow \underline{u}_1 < (1-\delta^T) S_1 < \delta S_1$. By Lemma 7, $u_\tau > \frac{v\tau}{1-\delta}$ for all $\tau = 1, ...T-2$ and $u_{T-1} = u_T = \frac{v(T-1)}{1-\delta}$. As we can increase u_1 up to $(1-\delta^T) S_1$ we obtain $u_{T-1} = \frac{vT}{1-\delta} - v$ and $u_T = \frac{vT}{1-\delta}$. If $u_1 \in \left[\underline{u}_1, (1-\delta^T) S_1\right]$, then we are done. Otherwise, to reach higher utility levels, $u_1 \in \left[(1-\delta^T) S_1, \delta S_1\right]$, we need to go to a higher T.

Now, we want to show that our support for $\tau < T$, where the buyers' follow a $T-$stage support, allows us to support the entire set of feasible payoffs and complete the proof.

Lemma 9 *Consider a T-stage support with (a) $u_\tau \geq \frac{v\tau}{1-\delta}$ for $\tau = 1, ...T-1$ and (b) $\frac{v(T-1)}{1-\delta} < u_T < \frac{vT}{1-\delta}$. Then, for any $\sigma \leq \tau$ and $\tau < T$, except for $\sigma = 0$ and $\tau = T-1$ then, if the support condition (4) holds at (σ, τ), then it holds at $(\sigma, \tau+1)$.*

Notice by Lemma 7 that conditions (a) and (b) of Lemma 9 are satisfied by our support sequence given our choice of T. From Lemma 9 the following corollary is immediate.

Corollary 10 *A $T-$stage support holds if it holds for state (τ, τ) for $\tau = 1, ...T$.*

Thus, we only need to check the support conditions with respect to cash-ins, $\sigma = \tau$.We now have the following proposition.

Proposition 11 *Consider a $T-$stage support and suppose that $\delta + \delta^{T-1} \geq 1$. Then, the support condition (4) holds for all $\sigma = 1, ...\tau$ and all $\tau \geq 1$ if $\left(1 - \delta^{T-1}\right) S_1 \leq u_1 \leq \delta S_1$. Thus, every $u_1 \in [0, \delta S_1]$ can be supported if $\delta \geq 1/2$.*

See Figure 2.

An immediate corollary is

Corollary 12 *In the limit, as $\delta \rightarrow 1$, the seller's minimum share of the surplus goes to zero.*

In the bargaining literature it is often the case that there is a mathematical equivalence between a model with a seller facing a single buyer whose type lies in a continuum and a seller who faces a continuum of agents where each one is a different type. This is because once an agent has accepted the offer, then they leave the game. This is not true in the model that we examine, since buyers do not leave the game once they accept a seller's offer. Since buyers remain in the game, their purchasing decisions can be implicitly and repeatedly coordinated over time. This can be seen by comparing our work with FLT (1985).

6 Delay and Inefficient Equilibria

Now, we show that equilibria do not necessarily have to be efficient. Due to Proposition **??**, every equilibrium is a $t-$cycle equilibrium. Thus, we examine equilibria when $t \geq 2$. On the equilibrium path, there are no sales in periods 1 through $t-1$, and then a sale of t units at a price p_t in period t. Thus, states $(1,0)$ through $(t-1,0)$, are "delay states." The continuation state after the sale is then $(t+1,t)$ which is equivalent to state $(1,0)$. The payoffs in a $t-cycle$ equilibrium are then $\pi_t = \frac{p_t}{1-\delta^t}$ for the seller, since he collects the revenues of p_t once every t periods, and $u_t = \frac{1}{1-\delta^t}\left[\frac{vt}{1-\delta} - p_t\right]$ for the buyers, since a purchase is made once every t periods and each purchase is for t units at a price p_t. Because of delay, the realized joint surplus in a $t - cycle$ equilibrium is less than the maximal surplus S_1. Furthermore, the continuation path in a $t - cycle$ equilibrium always has a smaller surplus than the efficient surplus for any state $(t,0)$. Letting Ψ_t be the realized joint surplus at the time of a sale, we have the equilibrium relationship

$$\pi_t + u_t = \frac{vt}{(1-\delta)(1-\delta^t)} \equiv \Psi_t,$$

Note that $\Psi_1 = \delta^{t-1}\Psi_t$, $\Psi_2 = \delta^{t-2}\Psi_t$ and so on for all delay states. The same pattern holds for seller profits and (incremental) buyer utilities.

To derive the equilibria, we must have delay and thus starting from a state $(1,0)$ we need to specify *approach* conditions for the equilibrium. The approach conditions mean that the seller finds it optimal to not offer any goods until the state is $(t,0)$. For the buyers, we must specify cut-off rules of when to accept offers of σ units up to period t. Given the prices that buyers are willing to accept, the seller must prefer to delay making such an offer until period t . Note that it is never credible for buyers to reject all seller offers due to flow dominance.

The buyers' cut-off rules in periods up to t are to reject any price greater than $p(\sigma,\tau)$, where $p(\sigma,\tau)$ satisfies

$$\frac{v\sigma(1-\delta^{t-\tau})}{(1-\delta)} + \delta^{t-\tau}\max\left[\frac{v\sigma}{(1-\delta)}, u_t\right] - \delta^{t-\tau}u_t \leq p(\sigma,\tau) \leq \frac{v\sigma}{(1-\delta)} + \delta^{t-(\tau-\sigma)}u_t, \tag{8}$$

where we have used $\delta^{t-(\tau-\sigma)}u_t = u(\tau+1-\sigma,0)$. The left hand side of (8) represents the difference in gross surplus for an individual buyer between buying the offer and rejecting it, since other buyers are expected to reject and hence he believes the state will be $(\tau+1,0)$. The first term is the buyers flow payoff of receiving σ units and the second terms represents his option of either not buying or buying with the other buyers, once the state reaches $(t,0)$. The right hand side represents the difference for an individual buyer between buying and not buying the package, given that all other buyers are expected to buy the package and the state will be $(t+1,\sigma)$. Note that we have used $\delta^{t-\tau}u_t = u(\tau+1,0)$ and $\delta^{t-(\tau-\sigma)}u_t = u(\tau+1-\sigma,0)$, since the first sale along the equilibrium path occurs in state $(t,0)$. Clearly, there exists a set of prices that satisfy (8). This is due to the coordination among buyers.

As with the analysis of the efficient path, we need to find a set of prices such that the seller would prefer not to deviate. There are, however, additional approach conditions for the seller in the inefficient equilibrium. In state $(\tau, 0)$ for $\tau < t$, it must be that the seller prefers the equilibrium path payoff of $\pi_\tau = \delta^{t-\tau}\pi_t$ to selling σ units in period τ at a price of $p = p(\sigma, \tau)$ and receiving a payoff of $p(\sigma, \tau) + \delta^{t-(\tau-\sigma)}\pi_t$. In other words, we must have

$$\delta^{t-\tau}(1-\delta^\sigma)\pi_t \geq p(\sigma, \tau) \text{ for } \sigma = 1, ..., \tau \text{ and } \tau = 1, ..., t-1. \tag{9}$$

We now provide the following lemma which greatly simplifies the buyer and seller approach conditions.

Lemma 13 *If the buyer and seller approach conditions, (8) and (9), hold for $\sigma = \tau$, at each $\tau = 1, ..., t-1$, then the conditions hold for all feasible pairs (σ, τ).*

P roof. Beginning with the seller approach conditions, (9), note that $\delta^{t-\tau}$ is strictly increasing in τ. Thus, we can set $p(\sigma, \tau)$=$p(\sigma, \sigma)$ and the condition holds for $\sigma < \tau$. Now, for the buyer approach conditions, (8), note that the right hand side is increasing in τ. We claim that the left hand side is decreasing in τ. When $\frac{v\sigma}{(1-\delta)} \geq u_t$, then the left hand side is $\frac{v\sigma}{(1-\delta)} - \delta^{t-\tau}u_t$, which is falling in τ. If $\frac{v\sigma}{(1-\delta)} < u_t$, then the left hand side is $\frac{v\sigma(1-\delta^{t-\tau})}{(1-\delta)}$, which is also strictly decreasing in τ. ■

Thus, the lemma shows that we need only find $t-1$ distinct prices, $p(1, 1), ..., p(t-1, t-1)$. Intuitively, it is sufficient to deter the seller from selling the maximum units as soon as possible, the "cash-in constraint".SOME ECONOMICS FOR THIS For example, if the seller does not offer one unit in state $(1, 0)$, then he will not be tempted to consider selling one unit in a later state.

Lemma 14 *In any inefficient $t-cycle$ equilibrium it is necessary that $u_t > 0$. Thus, buyers must receive positive utility.*

P roof. Note that when $u_t = 0$, for any deviation offer by the seller in period τ for σ units, the condition (8) immediately reduces to $p(\sigma, \tau) = \frac{v\sigma}{(1-\delta)}$. Thus, take $\sigma = \tau = t-1$ and consider condition (9). Since $\pi_t = \Psi_t$, when $u_t = 0$, we have

$$\delta(1-\delta^{t-1})\Psi_t \geq \frac{v(t-1)}{(1-\delta)} \Leftrightarrow \frac{1-\delta^t}{1-\delta} \geq t$$

after simplification. Clearly, this is always false for $t \geq 2$ and any δ. ■

The intuition for this result is quite simple. The buyers' approach condition at $u_t = 0$ says that the buyers will always accept any offer that gives them positive utility. Thus, the seller can always make sales that extract buyers in periods before period t by speeding up sales and increase his profits.

Combining the buyer and seller approach conditions, (8) and (9), we see that supporting prices exist if and only if

$$\frac{v\tau(1-\delta^{t-\tau})}{(1-\delta)} + \delta^{t-\tau}\max\left[\frac{v\tau}{(1-\delta)}, u_t\right] - \delta^{t-\tau}u_t \leq p(\tau,\tau) \leq \delta^{t-\tau}(1-\delta^{\tau})\pi_t \tag{10}$$

for $\tau = 1, ...t-1$.

Recalling $\pi_t + u_t = \Psi_t$, conditions (10) become

$$(\delta^{t-\tau} - \delta^t)(\Psi_t - u_t) \geq \frac{v\tau(1-\delta^{t-\tau})}{(1-\delta)} + \delta^{t-\tau}\max\left[\frac{v\tau}{(1-\delta)}, u_t\right] - \delta^{t-\tau}u_t \tag{11}$$

for $\tau = 1, ...t-1$. We now provide a sufficient condition on δ^t such that if δ^t is above a threshold, the approach conditions (11) are satisfied. We define $a(d) \equiv -\ln\left[-\frac{d\ln(d)}{1-d}\right]$, and note that there exists a unique root $d^* \in (0,1)$ for $d = a(d)$. Also, define

$$\underline{u}^A = \delta^t\Psi_t - \frac{v}{\ln\delta}\left(\frac{1}{1-\delta} + a(\delta^t)\right) \text{ and } \overline{u}^A = \Psi_t + \frac{v}{\delta^t\ln\delta}\left(\frac{1-\delta^t}{1-\delta}\right).$$

Lemma 15 *If $\delta^t > d^*$, then there exist $\underline{u}^A$ and $\overline{u}^A$ such that the approach conditions (11) are satisfied for any $u_t \in \left(\underline{u}^A, \overline{u}^A\right)$ in a $(t,0)$ equilibrium.*

Numerically, the equation root d^* is about .439. Thus, for $t = 2$, δ must be at least $\sqrt{.439}. = 0.663$ One could interpret the world as having two periods, period 1 and a period 2, where the discount factor for period 2 is δ^t. Hence, the longer delay in equilibrium, the higher must be δ so that the seller will not find a profitable deviation. Specifically, δ must exceed $d(t) \equiv \sqrt[t]{d^*}$ which is clearly increasing in t.

Lemma 15 provides a lower and upper bound on buyers' payoffs. The bounds on utility $\underline{u}^A$ and $\overline{u}^A$ depend on δ and t and they are derived in the Appendix. At d^*, $\underline{u}^A = \overline{u}^A$, and for all t and δ pairs where $\delta^t > d^*$, we have $\underline{u}^A < \overline{u}^A$. This can be seen graphically for $t = 2$ in the following figure.

As with efficient equilibria, we must ensure that players cannot do better by deviating in state $(t,0)$ as well as state $(\tau,0)$, for any τ greater than t. To show that a deviation cannot be profitable, we must specify the continuation payoffs from state $(\tau,0)$ and other "off-equilibrium path" states.

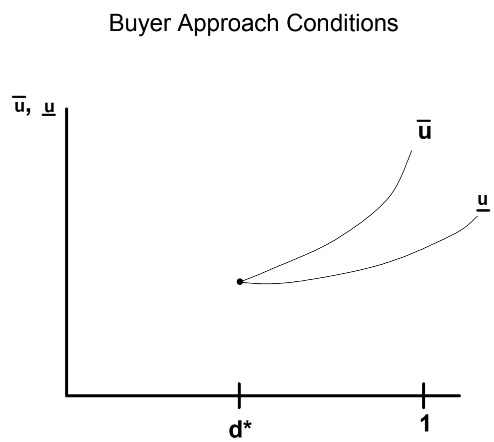

Figure 3:

As before we construct continuation payoffs so that in all states $(\tau, 0)$, the seller offers τ units at a price p_τ and this is accepted by all buyers. Thus, the continuation state is $(\tau+1, \tau)$, which returns the quality gap to 1; due to stationarity, this is equivalent to being back on the equilibrium path at $(1, 0)$.

The payoffs with a cash-in support at $(\tau, 0)$ satisfy $\pi_\tau = p_\tau + \delta\pi_1$ and the buyers' payoffs are $u_\tau - p_\tau + \delta u(\tau+1, \tau) = \frac{v\tau}{1-\delta} - p_\tau + \delta u_1$. Note, that from $(\tau, 0)$ the surplus on the continuation path is $\Psi_\tau = \frac{v\tau}{1-\delta} + \delta\Psi_1 = \pi_\tau + u_\tau$.

By analogy to the efficient $(1, 0)$ equilibrium, the support conditions for inefficient $t - cycle$ equilibrium can be derived via a cash-in support when the state is $(\tau, 0)$ where $\tau > t$. In particular buyer cut-off rules satisfy (2) and seller profit satisfy (3). We will use a particularly simple support for the off the equilibrium states. The purpose of this section is to demonstrate inefficient equilibria and not to characterize the entire set. Thus, we assume a constant support utility $\overline{u}$ for all states beyond $(t, 0)$; e.g. $(t+1, 0)$, $(t+2, 0)$...

Combining the buyer and seller conditions and incorporating the realized surplus, which differs from the maximal surplus in the efficient equilibrium, we obtain

$$\Psi_\tau - \delta\Psi_{\tau+1-\sigma} \geq \overline{u}(1-\delta) + g(\sigma, \overline{u}) \tag{12}$$

which must hold for all $\tau > t$ and all σ between 0 and $\tau - 1$.

Similar to Lemma **??** we can show equilibrium conditions (12) can be reduced to only two necessary and sufficient conditions.

Lemma 16 *An inefficient $t-$ cycle equilibrium is supported by a cash-in outcome if and only if (12) holds for (σ, τ) at $(0, t+1)$ and $(t+1, t+1)$;*

$$\Psi_{t+1} - \delta\Psi_{t+2} \geq \overline{u}(1-\delta) + g(0, \overline{u}) \tag{13}$$

$$\Psi_{t+1} - \delta\Psi_1 \geq \overline{u}(1-\delta) + g(t+1, \overline{u}) \tag{14}$$

The proof is analogous the proof of Lemma **??**. The conditions can be simplified to

$$\frac{vt}{1-\delta^t} + v - \frac{\delta v}{1-\delta} \geq (1-\delta)\overline{u} \tag{15}$$

$$u \geq \frac{1-\delta}{\delta^t}\overline{u} \tag{16}$$

Inequality (15), follows from (13), since $g(0, \overline{u}) = 0$. It demonstrates that the seller in period $t+1$ prefers to cash-in and sell $t+1$ units immediately rather than delay and sell $t+2$ in the following period. This is despite the fact that the surplus in $t+2$, Ψ_{t+2}, is larger than the surplus in $t+1$, Ψ_{t+1}. The condition depends on two components. First, Ψ_{t+1} exceeds $\delta\Psi_{t+2}$, reflecting the loss in revenue for the seller from delaying sales. Second, this revenue loss exceeds the benefit of lowering the buyers' utilities by $(1-\delta)\overline{u}$. Thus, the key to understanding the condition is that the loss in surplus must be large from not selling immediately relative to the savings of buyer payoff. As t gets large, the seller is more tempted to sell immediately since there is a greater surplus loss by not selling the units. Alternatively, for a given $\overline{u}$, the seller must have a sufficiently large number of units to sell to deter a delay deviation (when more units will be available for sale).

By direct comparison, the upper bound on $\overline{u}$ is below $\frac{v(t+1)}{1-\delta}$, since

$$t\delta^{t-1} < \frac{1-\delta^t}{1-\delta}$$

The left side is proportional to the surplus of receiving t units in t periods from the present, while the right hand side is proportional to the (interim) flow surplus of one unit over t periods, beginning today. NOTE: WHY IS THIS THE NECESSARY CONDITION. If $\overline{u}$ were to exceed $\frac{v(t+1)}{1-\delta}$, then the seller necessarily prefers to delay and let the surplus grow.

Condition (14) simplifies to (16), since $\overline{u} < \frac{v(t+1)}{1-\delta}$ implies that $g(t+1, \overline{u}) = \frac{v(t+1)}{1-\delta} - \delta\overline{u}$. This means that a deviating buyer who makes a purchase of $t+1$ units will not exercise the implicit option of buying subsequent seller offers. In other words, $\overline{u}$ is too small for such a buyer to remain active in the market. We now interpret the condition. A deviating buyer receives a surplus $\frac{v(t+1)}{1-\delta} - p$, if the seller offers a price greater than p_{t+1}. Since this surplus must be smaller than the value of waiting, $\delta\overline{u}$, and this must hold for any price above p_{t+1}, it must hold at the equilibrium price. The equilibrium price is related to the equilibrium buyer surplus via $\overline{u} = \left(\frac{v(t+1)}{1-\delta} - p_{t+1}\right) + \delta u_1$. Thus, upon noting that $u_1 = \delta^{t-1}u$, we see that an individual buyer's deviation payoff is simply $\overline{u} - \delta^t u$, reflecting the loss of all future surplus arising from further purchases. This is a coordination issue that is important in generating equilibria that is not applicable in the standard durable goods model where once a buyer makes a purchase he leaves the market.

Thus, we have found conditions for the approach states prior to state $(t, 0)$ and for the support for states $(t+1, 0)$ and beyond. It remains to find the equilibrium conditions for state $(t, 0)$: First, the seller can delay making a sale in state $(t, 0)$ and so we must have $\pi_t \geq \delta\pi_{t+1}$. Since $\Psi_\tau = \pi_\tau + u_\tau$, this profit condition is

$$\Psi_t - \delta\Psi_{t+1} \geq u_t - \delta\overline{u} \tag{17}$$

which simplifies to

$$\frac{vt}{1-\delta^t} - \frac{\delta v}{1-\delta} + \delta\overline{u} \geq u \tag{18}$$

Condition (**??**) says that the seller prefers to sell t units in period t, to delaying sales. It is analogous to condition (15), except for the fact that the buyers's surpluses changes from u in $(t, 0)$ to $\overline{u}$ in state $(t+1, 0)$ and that the seller has t rather than $t+1$ units to offer to buyers. Thus, rather than generating a strict upper bound, the condition puts an upper bound on the current payoff of u in terms of the future payoff of $\overline{u}$. A higher future payoff thus allows a higher current payoff, since the seller is more willing to avoid delay.

Second, just as the seller must not be able to raise the price in $(t+1, 0)$, the seller must not be able to raise the price from p_t at state $(t, 0)$. As before each buyer must find it optimal to reject any offer above p_t. This requires

$$g(t, \overline{u}) \leq p_t \leq \frac{vt}{1-\delta} + \delta u_t \tag{19}$$

which is analogous to condition (2). Using the fact that $u_t = \Psi_\tau - \frac{p_t}{1-\delta^t}$, (19) becomes

$$g(t, \overline{u}) \leq -(1-\delta^t)u_t + \frac{vt}{1-\delta} \tag{20}$$

since the right hand side of inequality (19) is always satisfied, due to $u_t > 0$. Condition (20) shows that buyers will accept any price p_t or below if all the other buyers accept, but will reject any price greater if all other buyers reject. Again, this an issue of buyer coordination, where buyers' purchasing decisions depends on what they expect others buyers to do.

To summarize, we have derived a system of four inequalities that must be satisfied for the outcome to be a cash-in at $(t, 0)$ and any state $(\tau, 0)$ (remind about translation states) for all $\tau > t$. They are conditions (15), (16), (**??**), and (20). A straightforward analysis of these inequalities yields the following result:

Lemma 17 *Suppose that $\delta > d(t)$. Then the set of buyer payoffs, u_t, that can be supported by a cash-in outcome at states $(\tau, 0)$, for $\tau \geq t$ is given by $0 \leq u_t \leq \Psi_t - \delta^2 S_1$.*

Figure 4 depicts the payoffs that can be supported in Lemma 17. Examining the figure demonstrates many points. First, a given $\overline{u}$ can be used to support a range of u_t values. For lower values of u_t, $\overline{u}$ must be relatively low, while higher values of u_t require a higher value of $\overline{u}$. As $\overline{u}$ goes to 0, u_t must necessarily go to 0; this is Lemma 14. The maximum buyers' utility is determined by only two of the constraint in Figure 4. One of these constraints is (15), which provides a strict upper bound on $\overline{u}$. When it binds, this constraint says that the seller is indifferent between selling $t+1$ units at price p_{t+1} and delaying till next period and selling $t+2$ units at price p_{t+2}. Given the maximum level to which $\overline{u}$ can be pushed, constraint (18) allows for the equilibrium utility u_t to be as large as possible. This constraint states that the seller is indifferent between the equilibrium of selling t at p_t and selling $t+1$ units at price p_{t+1}, when buyers receive $\overline{u}$ in states greater than $(t, 0)$. Thus, when the seller is pushed to indifference between sale and delay, the buyers' payoff is as large as possible. For a given $\overline{u}$, for u_t to be rising the buyers must be coordinating on a lower price, p_t, such that any price above p_t will be rejected. In the interior of the graph, below (18), the seller strictly prefers selling in $(t, 0)$ to delay.

Figure 4.

The final step to determining the set of equilibrium payoffs is to combine the approach and support conditions. This involves comparing $\underline{u}^A, \overline{u}^A$ and $\Psi_t - \delta^2 S_1$.

Proposition 18 *For any $t \geq 2$, if $\delta > d(t)$, then there exists a $t-$cycle equilibrium. The equilibrium range of buyer payoffs is given by: (i) If $\delta \in (d(t), \overline{d}\ (t))$, then $\underline{u}^A \leq u_t \leq \overline{u}^A$. (ii) If $\delta \in (\overline{d}\ (t), 1)$, then $\underline{u}^A \leq u_t \leq \Psi_t - \delta^2 S_1$.*

We show in the Appendix that $\Psi_t - \delta^2 S_1$ always exceeds $\underline{u}^A$ as long as δ is greater than $d(t)$. At relatively low δ, the approach conditions are more difficult to satisfy than the continuation support conditions. The intuition for this is that at a low δ, for any given buyer payoff the seller is not willing to delay a sale until relative temptation is to cash-in quickly rather than defer payoffs into the future. While for a higher δ this is not the case.

Proposition 19 *For any δ, the maximum buyer share of the equilibrium surplus is bounded below by* $\frac{\overline{u}^A}{\Psi_t} \geq 1 + \frac{(1-d^*)^2}{d^* \ln d^*} \cong .13$

Importantly, this bound is independent of δ. Seller does not have extraction power even when δ goes to 1.

7 Conclusions

To be done.

8 Appendix A- Preliminary Results.

The proof of Lemma 1.

P roof. Depending on the history, buyers may also hold a subset, possibly null, of units $\{Q+2, ..., t-1\}$. Without unit $Q+1$, a buyer who rejects the upgrade offer will receive a flow payoff of vQ and have the same quality holdings in period $t+1$. A buyer who accepts will receive a flow payoff of vt in period t and hold $\{1, ..., t\}$ next period. We need to show that accepting yields a strictly higher payoff than rejecting, for any strategy choices of other buyers and the seller following the upgrade offer.

Obviously, accepting yields a higher flow payoff in period t since $vt - p > vQ$. Thus, it remains to show that accepting cannot lead to a lower payoff in the continuation. This is a simple consequence of the upgrade payoff structure. Consider an arbitrary sequence of offers from period $t+1$ onward. Let us compare two buyers: B_1 holds units $\{1, ..., t\}$ and B_2 lacks one or more of these units. In choosing from the offer sequence, B_1 is always weakly better off than B_2 since both buyers choose from the same sequence and any acceptance choices of B_2 can be duplicated by B_1. For the same acceptance choices, both buyers make the same payments and the payoff comparison then reduces to the surplus flows from units that are held. But, given the same acceptance decisions, B_2 can never acquire a larger set of quality units than B_1. If B_2 ever receives a quality surplus flow vq with $q > t$ in a period $\tau \geq t+1$ then B_1 must also receive the same flow of vq in τ. This is because the flow payoff vq implies that B_2 holds units $\{1, ..., q\}$, but not unit $q+1$, and the units $\{t+1, ..., q\}$ must have been acquired through offer acceptances after period t and, therefore, are also held by B_1. Given their initial holdings in $t+1$, B_1 never has a quality flow surplus below vt while B_2 remains at the lower flow level of vQ until (and if) unit $Q+1$ is acquired. Thus, B_1 is always weakly better off than B_2 for any offer sequence.

It is now clear that every buyer will choose to accept the upgrade offer. Given any strategy choices of other buyers and the seller following the upgrade offer, a buyer who accepts always has a weakly larger payoff from $t+1$ onward, with respect to the continuation sequence of offers implied by the strategies, and a strictly larger flow payoff in period t. Note that we are assuming the continuation sequence of offers does not depend on the choice of a specific individual buyer to accept or reject the upgrade offer (as in GSW). ■

Proof of Lemma 2.

P roof. Proof: By Lemma 1, at the start of the game the seller can offer unit 1 for a price of $p_1 < v$ and every buyer will accept. Also, by Lemma 1, in period 2 when all buyers hold unit 1 the seller can offer unit 2 for a price of $p_2 < v$ and every buyer will accept. By induction, in any period t and for any history in which all buyers hold units $\{1, ..., t-1\}$ we can apply Lemma 1 to see that the seller can sell unit t for a price $p_t < v$. Each price can be arbitrarily close to v, so letting $v - \epsilon = p_t$ for all t, the seller's payoff from the start of the game must be at least $(v-\epsilon)(1+\delta+\delta^2+...) = (v-\epsilon)/(1-\delta)$. As this must hold for any $\epsilon > 0$, we are done.

For the continuation result, simply apply Lemma 1 with $p_t = (v-\epsilon)(t-Q)$ in period t and then apply Lemma 1 as above, starting in period $t+1$. ■

Proof of Lemma 3.

P roof. By stationarity, it is sufficient to prove the result for states of the form $(t, 0)$ since any state of the form (τ, Q) has the same quality increments and payments as $(\tau - Q, 0)$. Consider state $(t, 0)$ and a continuation path $(\tau, q_{\tau-1})$ for $\tau \geq t+1$. By Lemma 2, we know the seller's payoff in state $(t, 0)$ is positive. This implies that $q_{\tau-1} > 0$ for some τ. Otherwise, we have $q_{\tau-1} = 0$ for all τ and buyers must have a payoff of zero since they never acquire unit 1. But then buyer payments to the seller must be zero and, hence, the seller's payoff would be zero, which is not possible. Thus, $q_{\tau-1} > 0$ for some τ. Relabel so that τ denotes the first such period, so that $q_{\tau-1} > 0$ and $q_{\tau'-1} = 0$ for $\tau' < \tau$. Thus, the quality gap rises from t in state $(t, 0)$ to $\tau - 1$ in state $(\tau - 1, 0)$ and then goes to a gap of $\tau - q_{\tau-1}$, which is less than or equal to the previous gap of $\tau - 1$, in state $(\tau, q_{\tau-1})$

. By stationarity, the continuation path from $(\tau, q_{\tau-1})$ has the same quality increments as state $(\tau - q_{\tau-1}, 0)$. Thus, the quality gap will thereafter cycle repeatedly from size $\tau - q_{\tau-1}$ up to $\tau - 1$ and the continuation path from $(t, 0)$ has a bounded quality gap. ■

Proof of Proposition 4.

P roof. Starting from state $(1, 0)$, we know from Lemma 3 that the quality gap is bounded and, therefore, that there is a first date, say t, at which a sale involving unit 1 takes place. If $t = 1$, we are done as stationarity implies we have a 1-cycle equilibrium. So, consider $t > 1$. By construction, the maximal quality held by buyers before period t is zero, so the state is $(t, 0)$, and $q_t > 0$ results from sales in period t. A potential complication with state $(t, 0)$ that does not occur with $(1, 0)$ is that $(t, 0)$ corresponds to histories in which buyers acquired no quality units as well as histories in which they acquired some subset of $\{2, ..., t-1\}$. By definition, however, stationarity requires that the seller offer (s) in $(t, 0)$ and buyer acceptance choice(s) are the same across these histories since strategies only depend on the state $(t, 0)$.

Suppose that the sale at date t does not result in $q_t = t$ or, in other words, buyers do acquire the full feasible set of units $\{1, 2, ..., t\}$. This implies that, for some τ where $1 \leq \tau < t$, buyers acquire units $\{1, ..., \tau\}$ and they do not acquire unit $\tau + 1$. Also, let p denote the total payment made by a buyer to the seller for all bundles purchased in state $(t, 0)$. Finally, note that whether or not any of the units in $\{\tau + 2, ..., t\}$ are held by buyers before period t or acquired in t, the state in period $t + 1$ will be $(t + 1, \tau)$.

By construction, the equilibrium buyer continuation payoff from state $(t, 0)$ is given by

$$u(t, 0) = v\tau - p + \delta u(t + 1, \tau)$$

as the quality flow utility is $v\tau$ and the payment is p in $(t, 0)$, and next period's state is $(t + 1, \tau)$. We will show that a profitable deviation, namely, offering a bundle of τ units (units $1, ..., \tau$) for some price $\hat{p}$, exists for the seller in period $t - 1$. Note that this is feasible for the seller in period $t - 1$ since $\tau < t$.

Before proceeding with the main argument, we need to develop two properties of buyer payoffs. Stationarity implies that the equilibrium path will follow a cycle, since state $(t + 1, \tau)$ has the same quality gap as state $(t + 1 - \tau, 0)$. Thus, the maximal buyer quality remains at τ until period $t + \tau$, when the state reaches $(t + \tau, \tau)$, at which time the maximal buyer quality rises to 2τ and the cycle begins again. Stationarity also implies that a buyer only needs to make purchases in the states where maximal quality rises in order to achieve the equilibrium buyer payoff. As noted above, the history of play only matters to the extent that it impacts maximal buyer quality. Thus, the bundle(s) offered by the seller in any state of the form $(t + k\tau, k\tau)$, where $k = 1, 2...$, must, at a minimum, always include the next τ units of quality. In particular, this is true for the history where buyers hold exactly the first $k\tau$ quality units (and no other units), since the maximal quality for this history is $k\tau$. Thus, an individual buyer never needs to hold more than these units in order to be able to reach the next equilibrium path level of maximal quality via purchases in state $(t + k\tau, k\tau)$. Furthermore, such a buyer can always choose from the same offered bundle(s) and price as any other buyer. It follows directly that the continuation payoff of a buyer only depends on holding the current maximal quality and it is independent of whether the buyer holds higher but non-contiguous units. This is the first property of buyer payoffs that we will need.

The second property is that, in equilibrium, the seller only receives revenues in states of the form $(t + k\tau, k\tau)$, where $k = 1, 2...$ This follows by stationarity. In $(t + k\tau, k\tau)$, in equilibrium, the seller offer must include units $\{k\tau + 1, ..., k\tau + \tau\}$ and all buyers must acquire these units. Thus, no buyer will ever pay a positive price for any bundle in states $(1, 0)$ through $(\tau, 0)$, since only units in $\{2, ..., \tau\}$ can be offered by the seller in equilibrium and these units will necessarily be acquired in state $(t, 0)$ when buyers also acquire unit 1. The same logic then applies for the next τ units, and so on.

We now proceed with the main deviation argument. To keep things simple, let us first consider the case where the history for state $(t, 0)$ has buyers holding no quality units. For the seller deviation in period $t - 1$, choose the price $\hat{p}$ for the bundle of units $\{1, ..., \tau\}$ so that

$$\hat{u} \equiv v\tau - \hat{p} + \delta v\tau + \delta^2 u\left(t + 1, \tau\right) = \delta u\left(t, 0\right) + \epsilon,$$

for a small $\epsilon > 0$. Combining this with the earlier expression for $u\left(t, 0\right)$, we find that $\hat{p} = v\tau + \delta p - \epsilon$. We claim that in any equilibrium continuation after this offer, all buyers will accept. By symmetry of strategies, in response to this offer in state $(t - 1, 0)$, all buyers must either accept or reject. Suppose the buyer strategy calls for a rejection and consider the decision of an individual buyer. Because no other buyer accepts, the continuation state will be $(t, 0)$. By accepting, the payoff for an individual buyer is $\delta u(t, 0) + \varepsilon$. To see this, note first that the individual buyer receives a flow of $v\tau - \hat{p} = -\delta p + \epsilon$ in period $t - 1$. Now, consider period t. By making no purchase in period t, the buyer receives a flow of $v\tau$. Finally, consider the continuation state $(t + 1, \tau)$ following period t.

A complication is that, in addition to the first τ units, the outcome in state $(t, 0)$ may involve buyers acquiring units in $\{\tau + 2, ..., t\}$. By making no purchases in period t, the deviating buyer will lack these units in the future while other buyers possess them. But, as we showed above, this is of no consequence in a stationary equilibrium: a buyer holding exactly τ units obtains the same continuation payoff of $u(t+1,\tau)$.

Now adding the terms in periods $t-1$, t, and $t+1$ for a deviating buyer, we arrive at $\hat{u}$ as in the above equation. Thus, accepting the seller's deviation offer in period $t-1$ for τ units results in a higher payoff than rejecting and waiting whenever $\varepsilon > 0$. Thus, all buyers rejecting the offer is never an equilibrium continuation. In a symmetric equilibrium, it must be that all buyers accept the offer in $t-1$.

Now, to see that the deviation is profitable for the seller, note that the payoff to the deviation offer in period $t-1$ (where all buyers accept) is

$$\widehat{\pi} = \widehat{p} + \delta\pi(t,\tau) = v\tau + \delta p - \varepsilon + \delta^2\pi(t+1,\tau) = v\tau - \varepsilon + \delta\pi(t,0) > \delta\pi(t,0),$$

where we have used the definition of $\hat{p}$ and the equilibrium hypothesis for $(t,0)$, which implies $\pi(t-1,0) = \delta\pi(t,0)$ and $\pi(t,0) = p + \delta\pi(t+1,\tau)$. Thus, we cannot have $\tau < t$ in equilibrium.

Finally, we must verify that the same deviation will work for the seller when the history for state $(t,0)$ has buyers holding quality units (but not unit 1). By stationarity, seller's payoffs $\pi(t,\tau)$ and $\pi(t+1,\tau)$ are independent of these holdings. The only remaining possible complication is that the deviation offer in $t-1$ sacrifices revenues that would otherwise have been received by the seller from an offer of units in $\{2,...,t-1\}$. Stationarity, however, rules out any such revenues for the seller as we showed above. ■

9 Appendix B - Efficient Equilibria.

The following two lemmas are preliminary arguments to establish Lemma 6.

Lemma 20 *Consider a support where $u_\tau = \overline{u}$ for all $\tau \geq T$. Then if the support holds at τ for $\sigma = (\tau+1-T)+1$ to $\sigma = \tau$, then it holds at $\tau+1$ for $\sigma = (\tau+1+1-T)+1$ to $\sigma = \tau+1$.*

P roof. For T, the σ range of $(\tau+1-T)+1$ to τ is the range where the continuation state $(\tau+1-\sigma)$ is in the set of $\{1,...T-1\}$ with utility payoffs of $\{u_1,...u_{T-1}\}$. For (σ,τ) we have a valid support if and only if

$$S_\tau - \delta S_{\tau+1-\sigma} \geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1})$$

which holds if and only if

$$v\tau + \frac{\delta v\sigma}{1-\delta} \geq \overline{u} - \delta u_{\tau+1-\sigma} + g(\sigma, \overline{u})$$

For $(\sigma+1, \tau+1)$ we have a valid support if and only if

$$v(\tau+1) + \frac{\delta v(\sigma+1)}{1-\delta} \geq \overline{u} - \delta u_{\tau+1-\sigma} + g(\sigma+1, \overline{u})$$

At any τ, there are exactly $(T-1)$ of the conditions where the continuation payoffs are $u_1, ... u_{T-1}$. We claim that (σ, τ) condition holding implies that $(\sigma+1, \tau+1)$ condition holds. It is sufficient to show that

$$\frac{v}{1-\delta} - g(\sigma+1, \overline{u}) \geq -g(\sigma, \overline{u})$$

which is valid if and only if

$$\frac{v}{1-\delta} \geq g(\sigma+1, \overline{u}) - g(\sigma, \overline{u}) \tag{21}$$

There are three cases that we need to check.

1) If $\overline{u} \leq \frac{v\sigma}{1-\delta}$, then $g(\sigma+1, \overline{u}) = \frac{v(\sigma+1)}{1-\delta} - \delta\overline{u}$ and $g(\sigma, \overline{u}) = \frac{v\sigma}{1-\delta} - \delta\overline{u}$, hence (21) is an equality.

2) If $\frac{v\sigma}{1-\delta} < \overline{u} \leq \frac{v(\sigma+1)}{1-\delta}$, then $g(\sigma+1, \overline{u}) - g(\sigma, \overline{u}) = \frac{v(\sigma+1)}{1-\delta} - \delta\overline{u} - v\sigma$, which is less than $\frac{v}{1-\delta}$ if and only if $\frac{v\sigma}{1-\delta} < \overline{u}$.

3) If $\frac{v(\sigma+1)}{1-\delta} < \overline{u}$, then $g(\sigma+1, \overline{u}) - g(\sigma, \overline{u}) = v(\sigma+1) - v\sigma < \frac{v}{1-\delta}$. ■

Lemma 21 *Consider a support where $u_\tau = \overline{u}$ for all $\tau \geq T$. Then, if support holds at τ and $\sigma = 0$, it holds for τ and all $\sigma = 1, ... \tau + 1 - T$.*

P roof. Given τ, consider support at $\sigma = 0, ... \tau + 1 - T$. Since $\sigma \leq \tau + 1 - T$ if and only if $T \leq \tau + 1 - \sigma$, we have

$$S_\tau - \delta S_{\tau+1-\sigma} \geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1})$$

if and only if

$$v\tau + \frac{\delta v\sigma}{1-\delta} \geq \overline{u}(1-\delta) + g(\sigma, \overline{u})$$

At $\sigma = 0$, we have $g(0, \overline{u}) = 0$, thus $(\sigma = 0, \tau)$ condition is $v\tau \geq \overline{u}(1-\delta)$.

Now consider $\sigma = 1, ... \tau + 1 - T$. We have two cases.

1) If $\overline{u} \leq \frac{v\sigma}{1-\delta}$, then support at σ requires

$$v\tau + \frac{\delta v\sigma}{1-\delta} \geq \overline{u}(1-\delta) + \frac{v\sigma}{1-\delta} - \delta\overline{u}$$

which holds if and only if

$$v(\tau - \sigma) \geq (1 - 2\delta)\overline{u},$$

which always holds if $\tau \geq \sigma$ and $\delta \geq 1/2$.

2) If $\overline{u} > \frac{v\sigma}{1-\delta}$, then support at σ requires

$$v\tau + \frac{\delta v\sigma}{1-\delta} \geq \overline{u}(1-\delta) + v\sigma,$$

since $\max\{\frac{v\sigma}{1-\delta}, \overline{u}\} = \overline{u}$. This holds if and only if

$$v\tau + v\sigma\left(\frac{\delta}{1-\delta} - 1\right) \geq \overline{u}(1-\delta),$$

but $v\tau \geq \overline{u}(1-\delta)$ by support at $\sigma = 0$, and $\frac{\delta}{1-\delta} \geq 1$ if $\delta \geq 1/2$. ■

Now, we are in position to prove Lemma 6.

P roof. From Lemma 20, we know that if support holds at T for $\sigma = 2, ...T$ it holds at $T+1$

for $\sigma = 3, ...T+1$ and for all $\tau > T$. From Lemma 21, we know for each $\tau \geq T$, that support at $\sigma = 0$ is sufficient for $\sigma = 1, ...\tau + 1 - T$. Now, $(\sigma = 0, \tau)$ implies $(\sigma = 0, \tau + 1)$, since

$$\sigma = 0, \tau \text{ is } v\tau \geq u_\tau - \delta u_{\tau+1} + g(0, u_{\tau+1}) = (1-\delta)\overline{u}$$

and

$$\sigma = 0, \tau + 1 \text{ is } v(\tau+1) \geq u_{\tau+1} - \delta u_{\tau+2} + g(0, u_{\tau+2}) = (1-\delta)\overline{u}.$$

Thus, the proof is complete, since if it holds at T it holds at $T+1$ and there after. ■

The proof of 7.

P roof. Let $\underline{u}_1 \equiv \frac{v}{1-\delta}\left[\frac{1-\delta^{T-1}}{1-\delta}\right]$ and $\underline{u}_\tau = v\tau + \delta u_{\tau+1}$ for $\tau = 1, ...T$. We first show that $\underline{u}_\tau \equiv \frac{v}{1-\delta}\left[\frac{1-\delta^{T-\tau}}{1-\delta} + (\tau - 1)\right]$ for $\tau = 1, ...T$. The proof holds via induction. Clearly, $\underline{u}_1$ applies by construction. We assume that it holds for τ and show that it holds for $\tau + 1$. By construction

$$\underline{u}_{\tau+1} = \frac{1}{\delta}\left[\underline{u}_\tau - v\tau\right] = \frac{1}{\delta}\left[\frac{v}{1-\delta}\left(\frac{1-\delta^{T-\tau}}{1-\delta} + (\tau - 1)\right) - v\tau\right] =$$

$$\frac{v}{\delta(1-\delta)^2}\left[\delta + \delta\tau - \delta^{T-\tau} - \delta^2\tau\right] = \frac{v}{1-\delta}\left[\frac{1-\delta^{T-(\tau+1)}}{1-\delta} + \tau\right].$$

Next, we show that (ii) $\underline{u}_\tau \geq \frac{v\tau}{1-\delta}$ if and only if $\delta \geq \delta^{T-\tau}$. Using the definition of $\underline{u}_\tau$

$$\frac{v}{1-\delta}\left[\frac{1-\delta^{T-\tau}}{1-\delta} + (\tau - 1)\right] \geq \frac{v\tau}{1-\delta} \text{ iff}$$

$$\frac{1-\delta^{T-\tau}}{1-\delta}+(\tau-1)\geq\tau \text{ iff } \delta\geq\delta^{T-\tau}$$

Finally, we show that

$$u_\tau=\frac{1}{\delta^{\tau-1}}\left[u_1-\frac{v}{1-\delta}\left[\frac{1-\delta^{\tau-1}}{1-\delta}+(\tau-1)\delta^{\tau-1}\right]\right]$$

We do this by induction. Clearly, it holds at $\tau=1$. Assume that it holds at τ and show that it holds at $\tau+1$.

$$u_{\tau+1}=\frac{1}{\delta}\left[u_\tau-v\tau\right]=\frac{1}{\delta^\tau}\left[u_1-\frac{v}{1-\delta}\left(\frac{1-\delta^{\tau-1}}{1-\delta}-(\tau-1)\delta^{\tau-1}\right)\right]-\frac{v\tau}{\delta}=$$

$$\begin{aligned}&\frac{1}{\delta^\tau}\left[u_1-\frac{v}{1-\delta}\left(\frac{1-\delta^{\tau-1}}{1-\delta}-(\tau-1)\delta^{\tau-1}+\tau\delta^{\tau-1}(1-\delta)\right)\right]\\=&\frac{1}{\delta^\tau}\left[u_1-\frac{v}{1-\delta}\left(\frac{1-\delta^\tau}{1-\delta}-\tau\delta^\tau\right)\right]=u_{\tau+1}\end{aligned}$$

The rest of the lemma follows by simple substitutions. ■

The proof of Lemma 9.

P roof. The condition for (σ,τ) is

$$S_\tau-\delta S_{\tau+1-\sigma}\geq u_\tau-\delta u_{\tau+1-\sigma}+g(\sigma,u_{\tau+1})$$

which holds if and only if

$$v(\tau-\sigma)+\frac{\delta v\sigma}{1-\delta}\geq u_\tau-\delta u_{\tau+1-\sigma}, \tag{22}$$

since $u_{\tau+1}>\frac{v\sigma}{1-\delta}$ for any $\sigma\leq\tau$ for $\tau<T$.

At $(\sigma,\tau+1)$, we need

$$S_{\tau+1}-\delta S_{\tau+2-\sigma}\geq u_{\tau+1}-\delta u_{\tau+2-\sigma}+g(\sigma,u_{\tau+2})$$

which holds if and only if

$$v(\tau-\sigma+1)+\frac{\delta v\sigma}{1-\delta}\geq u_{\tau+1}-\delta u_{\tau+2-\sigma}, \tag{23}$$

since $u_{\tau+2}>\frac{v\sigma}{1-\delta}$ for any $\sigma\leq\tau<T$. Using the definitions of $u_{\tau+1}$ and $\delta u_{\tau+2-\sigma}$, (23) becomes

$$v(\tau-\sigma+1)+\frac{\delta v\sigma}{1-\delta}\geq\frac{1}{\delta}(u_\tau-v\tau)-u_{\tau+1-\sigma}+v(\tau+1-\sigma)$$

which is equivalent to

$$v\tau+\frac{\delta^2v\sigma}{1-\delta}\geq u_\tau-\delta u_{\tau+1-\sigma}$$

Thus, all we need to show is that $v\tau + \frac{\delta^2 v\sigma}{1-\delta} \geq v(\tau - \sigma) + \frac{\delta v\sigma}{1-\delta}$, which holds for any $\delta \in [0,1]$ and any non-negative σ. ■

Proof of Propositions 11.

P roof. By Corollary 10, we need only check the conditions at (τ, τ) for $\tau = 1, ...T$. It is easy to show that at $\tau = 1$, requires $\delta S_1 \geq u_1$.

So, consider (τ, τ). First, assume $\tau \leq T - 1$. Since $u_{\tau+1} > \frac{v\tau}{(1-\delta)}$, $g(\tau, u_{\tau+1}) = v\tau$. Thus, the equilibrium support condition (4) becomes

$$\frac{\delta v\tau}{1-\delta} + \delta u_1 \geq u_\tau.$$

We claim that condition (τ, τ) implies condition $(\tau + 1, \tau + 1)$ for $\tau \leq T - 2$. In other words, we claim that $\frac{\delta v\tau}{1-\delta} + \delta u_1 \geq u_\tau$ implies $\frac{\delta v(\tau+1)}{1-\delta} + \delta u_1 \geq u_{\tau+1}$. Recall that $u_{\tau+1} = \frac{1}{\delta}(u_\tau - v\tau)$. So, condition $(\tau + 1, \tau + 1)$ can be written at

$$\frac{\delta^2 v(\tau + 1)}{1-\delta} + \delta^2 u_1 + v\tau \geq u_\tau.$$

Thus, it is sufficient to show that $\frac{\delta^2 v(\tau+1)}{1-\delta} + \delta^2 u_1 + v\tau > \frac{\delta v\tau}{1-\delta} + \delta u_1$. But, this holds if and only if $\delta S_1 + \frac{v\tau}{\delta} > u_1$, which is always the case for $\tau \geq 1$.

Thus, we are left with the (T, T) condition. Condition (T, T) becomes

$$\delta u_1 \geq (1 - \delta)u_T,$$

since $g(T, u_{T+1}) = \frac{vT}{1-\delta} - \delta u_T$ by Lemma 7. Thus, it is sufficient to show

$$\frac{\delta}{1-\delta} u_1 \geq \frac{1}{\delta}\left[\frac{\delta v(T-1)}{1-\delta} - (T-1)v + \delta u_1\right] \tag{24}$$

since condition $(T - 1, T - 1)$ holds if

$$\frac{\delta v(T-1)}{1-\delta} + \delta u_1 \geq u_{T-1} = v(T-1) = \delta u_T,$$

which holds if and only if

$$\frac{1}{\delta}\left[\frac{\delta v(T-1)}{1-\delta} - (T-1)v + \delta u_1\right] \geq u_T$$

Thus, condition (24) holds if and only if

$$\delta u_1 \geq v(T - 1).$$

From $u_1 \geq (1 - \delta^{T-1})S_1$, it is sufficient to show that $\frac{\delta(1-\delta^{T-1})S_1}{v} \geq T - 1$. At $T = 1$, this reduces to $0 = 0$, but $T = 1$ does not need a support beyond $(1, 1)$ so it is not an issue. Now, we do an induction. Assume for T with $\delta + \delta^{T-1} > 1$ and show it holds for $T+1$ with $\delta + \delta^{T} > 1$. So, we want to show that $\frac{\delta(1-\delta^{T})S_1}{v} \geq T$. Or,

$$\delta(1 + ... + \delta^{T-1}) > T(1-\delta).$$

We have at condition T $\frac{\delta(1-\delta^{T-1})S_1}{v} \geq T-1$, which holds if and only if

$$(1-\delta) + \delta(1 + ... + \delta^{T-2}) > T(1-\delta).$$

But,

$$\delta(1 + ... + \delta^{T-1}) > (1-\delta) + \delta(1 + ... + \delta^{T-2})$$

If and only if $\delta + \delta^{T} > 1$, which establishes the induction. Thus, the $T-$stage support holds. ■

10 Appendix C - Inefficient Equilibria.

The proof of Lemma 15.

P roof. In the conditions (11), τ assumes integer values $1, ..., t-1$. We will replace τ with a continuous variable, x, that assumes values in the interval $[0, t]$. This greatly simplifies the derivation of the sufficiency condition. It is useful to define three functions:

$$A(x, u, \delta, t) \equiv (\delta^{t-x} - \delta^{t}) \left[\frac{t}{(1-\delta)(1-\delta^{t})} - u\right]$$

$$B(x, \delta, t) \equiv \frac{vx}{1-\delta}(1 - \delta^{t-x})$$

$$C(x, u, \delta, t) \equiv \frac{vx}{1-\delta} - \delta^{t-x}u,$$

where $u \equiv u_t$. For (δ, t), take u. In terms of x, the conditions (11) become

$$\begin{aligned} A(x, u, \delta, t) &\geq B(x, \delta, t) \; for \; 0 < x \leq \frac{(1-\delta)u}{v} \\ and \; A(x, u, \delta, t) &\geq C(x, u, \delta, t) \; for \; \frac{(1-\delta)u}{v} < x \leq t. \end{aligned}$$

First, we find conditions when $A(x, u, \delta, t) \geq B(x, \delta, t)$ for all x in the interval $\left(0, \frac{(1-\delta)u}{v}\right]$. For convenience, let θ be defined as $\frac{(1-\delta)u}{v}$. We find a necessary and sufficient condition to hold. Suppressing arguments $A = B$ when $x = 0$. A is increasing and convex in x and equals 0 when x, while B is strictly concave in x and equal 0 when x equals 0. Thus, if at $x = 0$, $\frac{\partial A}{\partial x} \geq \frac{\partial B}{\partial x}$, then $A \geq B$ must all positive x. Calculating the partial derivatives, this yields the condition

$$\frac{v}{(1-\delta)} \left[\frac{t}{1-\delta^{t}} - \frac{1-\delta^{t}}{(-\ln\delta)\delta^{t}}\right] \geq u. \tag{25}$$

Next, suppose $\theta \leq x \leq t$. We find conditions when $A(x, u, \delta, t) \geq C(x, u, \delta, t)$ for all x in the interval $[\theta, t]$. This condition simplifies to

$$\delta^t \theta \geq x - \frac{t(\delta^{t-x} - \delta^t)}{1 - \delta^t} \equiv h(x, t, \delta). \tag{26}$$

$h(x, t, \delta)$ is strictly concave and equals 0 at $x = 0$ and $x = t$. Thus, $h(x, t, \delta)$ has a unique interior maximum at some $x^*(\delta, t)$ which is implicitly defined by $\delta^{x^*}(1-\delta^t) = (-\ln \delta^t)\delta^t$. Note that condition (25) implies that $\delta^t > \frac{\partial h}{\partial x}$ at $x = 0$.

PICTURE

In order for conditions (25) and (26) to hold,

$$\overline{u} \equiv \frac{v}{(1-\delta)} \left[\frac{t}{1-\delta^t} - \frac{1-\delta^t}{(-\ln \delta)\delta^t} \right] \geq u \geq \frac{vh(x^*, t, \delta)}{\delta^t (1-\delta)} \equiv \underline{u}. \tag{27}$$

must be satisfied. If the right hand side of (27) fails, then there exist a deviation by the seller of x greater than θ that will be profitable.

Now, we provide a condition such that $\overline{u} > \underline{u}$ and thus an equilibrium support condition exists. Comparing $\overline{u}$ and $\underline{u}$ and simplifying with the definition x^* we find that $\overline{u} > \underline{u}$ if and only if

$$\delta^t > -\ln \left[\frac{-\ln(\delta^t)\delta^t}{1-\delta^t} \right]. \tag{28}$$

Let y be defined as δ^t and set (28) as an equality. Then there exists a unique root in $(0, 1)$, since the left hand is linear and increases from 0 to 1 as y goes from 0 to 1, while and the right hand side is strictly decreasing in y, and approaches 0 as y goes to 1, (L'Hospital Rule). ■

P roof. Proof of Proposition 18. First, we demonstrate that for relatively small δ, $\overline{u}^A < \Psi_t - \delta^2 S_1$, while for higher $\delta's$ the converse is true. Comparing, we have $\overline{u}^A < \Psi_t - \delta^2 S$ if and only if

$$0 < 1 - \delta^t + \delta^t f(\delta), \tag{29}$$

where $\frac{\delta^2 \ln \delta}{1-\delta} \equiv f(\delta)$. One can show that $f(\delta)$ is negative, decreasing, and concave with limiting values of 0 as $\delta \to 0$ and -1 as $\delta \to 1$. One can then apply an induction argument for $t \geq 2$, to show that $\delta^t f(\delta)$ has the same properties. Note that $1 - \delta^t$ is positive, decreasing, and concave in δ with limiting values of 1 as $\delta \to 0$ and 0 as $\delta \to 1$. Thus, $1 - \delta^t + \delta^t f(\delta)$ is decreasing and concave and equals 0 at a unique $\delta \in (0, 1)$ which we define as $\overline{d}(t)$. To verify that $\overline{u}^A < \Psi_t - \delta^2 S_1$ at $\delta = d^*$, we note that

$$1 - \delta^t + \delta^t f(\delta) > 1 - 2\delta^t > 1 - 2d^* > 0.$$

The first inequality follows from $f(\delta) > -1$, the second by $\delta^t > d^*$, and the third by $d^* < 1/2$. Thus, we have established that $d(t) < \overline{d}(t) < 1$ and that $\overline{u}^A$ crosses $\Psi_t - \delta^2 S_1$ from below only once at $\delta = \overline{d}(t)$.

Next, we establish $\underline{u}^A < \Psi_t - \delta^2 S_1$ for all $\delta > d(t)$. Comparing, $\underline{u}^A < \Psi_t - \delta^2 S_1$ if and only if

$$0 < 1 + \delta^t f(\delta) - a(\delta^t). \tag{30}$$

From above, we characterized $\delta^t f(\delta)$. We can demonstrate that $-a(\delta)$ is negative, increasing, and concave with limiting value of 0 as $\delta \to 1$. Thus, the limiting value of $1 + \delta^t f(\delta) - a(\delta^t)$ as $\delta \to 1$ is 0. Note that since $\underline{u}^A = \overline{u}^A$ at $\delta = d(t)$, we then have, from above, $\Psi_t - \delta^2 S_1 > \overline{u}^A = \underline{u}^A$ at $\delta = d(t)$. By concavity of the component functions, the result is established. ■

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