Bundled Discounts by Independent Producers of Vertically Differentiated Goods.*

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Abstract

This paper studies the competitive effects of bundled discounts in a setting where the component goods are vertically differentiated and sold by otherwise unrelated firms. When firms decide simultaneously about their participation in a discounting scheme, in equilibrium, both pairs of firms offer bundled discounts and, relative to the no-bundling benchmark: (*i*) all headline prices rise; (*ii*) all bundle prices, net of the discount, rise; and (*iii*) all firms earn higher profits. Furthermore, the equilibrium corresponds to the worst scenario in terms of consumer and social welfare, when compared to bundled discounts only offered by a single pair of firms or to the no-bundling benchmark.

Keywords: Bundled Discounts, Bilateral Bundling, Vertical Differentiation.
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1 Introduction

Bundled discounts provide purchasers the opportunity to pay less for a bundle of products than the sum of the prices of the bundle's constituent products when purchased separately. These discounting schemes thus confront consumers with the choice between meeting all their requirements by buying a package at a discounted price and *à la carte* offerings.

Examples of companies offering bundled discounts include fast food restaurants, telephone companies, book stores, grocery stores and gasoline retailers, to name a few. Despite the fact that bundled discounts are a widespread business practice, the academic literature has devoted limited attention to this issue. So, little is known as to whether this type of discounting schemes should raise anticompetitive concerns.[1]

Bundled discounts affect the market outcome in at least two important ways. On the one hand, they can change the market structure by affecting firms' incentives to enter or exit the market. On the other hand, they introduce an additional instrument that enables some degree of price discrimination and, consequently, may have an important impact on both consumer surplus and social welfare. Accordingly, one can divide the literature on bundled discounts in two related strands. The first strand has investigated the case in which the bundled discount is offered by a multiproduct firm offering two or more goods or services as a package for a lower price than the aggregate price of its constituent parts. In particular, by examining a setting wherein a monopoly seller in one market faces competition in a second market, a set of recent models (e.g. Peitz (2008), Greenlee *et al.* (2008), Nalebuff (2005) and Nalebuff (2004)) has shown that the use of bundled discounts can lead to the exclusion of an existing (or potential) equally efficient rival that does not offer an equally diverse group of products.[2] The second strand of the literature analyzes the implications of bundled discounts, per se, on consumer surplus and on social welfare and dates back to Matutes and Regibeau (1992). These authors analyze the behavior of duopolistic firms, both

[1] As pointed out by Nalebuff (2005, p.364), "[t]he practice of bundle discounts is prevalent, but their effects on competition are not well understood." Along these lines, Kobayashi (2005) highlights that his "review of the economic literature generally confirms the US Solicitor General's view in *3M v. LePage*'s regarding the underdeveloped state of the economics literature and its position that the US Supreme Court should defer promulgation of antitrust standards for bundling."

[2] As highlighted in OECD (2008, p.25), the key issue analyzed by this strand of the literature is that "[b]y bundling together different products, a firm with market power may be able to offer the bundle at a price which is still profitable for the dominant supplier, but could make it impossible for rivals to match or better these offers. As a result, a rational profit-maximizing buyer is 'forced' to take the bundle, since good competitive offers are not available." Hence, "an equally or even more efficient firm making only one product or a subset of products in a bundle may not be able to match an aggregated discount".

supplying the two necessary components to make up a system. In particular, they study the incentives these firms have with respect to: (*i*) making the components compatible with those produced by the rival firm; and (*ii*) offering a discount to those consumers who decide to purchase the whole system from them. In their setting, the four possible combinations that make up the system are located at the corners of a unit square where consumers are uniformly distributed. The main conclusions are that, in most cases, firms will produce compatible goods and offer discounts to those consumers who purchase both components from them. However, there is a prisoner's dilemma in the sense that firms would have a higher payoff if they could commit not to offer discounts. In addition, when in equilibrium all firms offer discounts, welfare is lower than when there are no discounts.

This pathbreaking work by Matutes and Regibeau (1992) has been extended in several ways. In particular, Gans and King (2006), the most similar previous work to ours, investigate the case where each of the two components is produced by two single product independent firms.[3,4] More specifically, they study a setting where pairs of retailers, such as supermarket and retail gasoline chains, may offer bundled discounts to consumers who buy their products, with discounts being a fixed amount off the headline (or stand-alone) prices that partner brands continue to set independently. They show that when a bundled discount is offered, the headline prices of the two otherwise completely unrelated products become related, thus generating an externality in pricing between the two products. In their setting, unilateral bundling, i.e. bundling by each pair of firms assuming the other firms are not involved in bundling, increases the profits of the partner firms to the detriment of the remaining non-bundling firms. Nevertheless, if both pairs offer bundled discounts, then each firm's profits (and output) are *the same* as in the case where there are no bundled discounts. Moreover, bilateral bundling, i.e., bundling by both pairs of independent firms, leads to a social-welfare reduction, as some consumers simply find themselves consuming a sub-optimal branding mix in the sense that consumers end up wasting surplus by purchasing from firms in less desirable 'locations.'[5]

[3] Also, Thanassoulis (2007) builds on Matutes and Regibeau (1992) by introducing consumers who only value one of the components and by making a distinction between firm-specific and product-specific preferences. As in Matutes and Regibeau (1992) there is a prisoners' dilemma in the sense that firms lose from mixed bundling.

[4] Other work that focuses on bundled discounts by independent producers is Maruyama and Minamikawa (2009), who study the incentives for vertical integration.

[5] In recent empirical study using a detailed dataset regarding an Australian market - the Perth metropolitan area -, Wang (2009) finds that the Gans and King (2006) model captures the gasoline pricing behavior of those retail stations that are allied with, but are not operated by, supermarkets.

A common assumption of Matutes and Regibeau (1992), Gans and King (2006), Thanassoulis (2007) and Peitz (2008) is that products are horizontally differentiated.[6] However, there are several examples of bundled discounts in industries where, at least with respect to one of the products in the bundle, differentiation is clearly vertical. For instance, many supermarkets and discount stores in the U.S. and other countries (e.g. U.K., Australia, Portugal) offer a grocery-gasoline bundled discount whereby customers receive a discount on their grocery-gasoline purchases from the firms involved in the discounting scheme. It should be noted, however, that in this grocery-gasoline case, there is vertical differentiation, at least, on the groceries' side: most consumers consider that supermarkets chains and discount stores offer products and services of different quality. Another case in point refers to bundled discounts offered to consumers who look for airline tickets and car-rental, a common practice involving most airline companies in the world along with car renting firms. Low cost airlines and national carriers are also, in most cases, vertically differentiated. To the best of our knowledge, however, vertical differentiation has been neglected by the extant literature on bundled discounts.

In this paper we contribute to cover this gap in the literature by proposing a model of strategic interaction between four producers of two different and unrelated products to study the likely competitive effects of bundled discounts in the presence of vertical differentiation. In our setting, consumers are arrayed on a unit square but, differently from Gans and King (2006), each axis measures consumers' valuations for quality of the two products. Each good is produced by two firms, a high quality producer and a low quality producer. Pairs of firms may then agree to offer jointly a bundled discount across the two products and to share the costs of that discount. We address, apart from the no discounting benchmark case, three different scenarios: (*i*) unilateral bundled discount by the low quality firms; (*ii*) unilateral bundled discount by the high quality firms; and (*iii*) bilateral bundling.[7]

This theoretical framework enables us to raise a number of interesting questions: What are the welfare effects of bundled discounts in each of the three scenarios described above? Are bundled discounts consumer-surplus-enhancing? Should bundled discounts be free of antitrust concerns in this context? Which firms have the highest incentives to offer the

[6] Also, Caminal and Claici (2007) use a two-period horizontal differentiation model in which firms are able to discriminate between first time and repeat buyers. In the authors' opinion, "[l]oyalty programs can perhaps be interpreted as a form of price discrimination analogous to quantity and bundled discounts."

[7] We assume that only firms of the same quality level may offer bundled discounts. The motivation for this assumption is that a producer offering a high quality product is probably not interested in being allied with a low quality producer (of the other good) as this may seriously hurt the reputation of its firm. An advantage of this assumption is that it allows to keep the equilibrium analysis tractable.

discounts? Does the prisoners' dilemma identified in the literature carry over to the case of vertical differentiation? The answer to this and other related questions is, to our knowledge, not yet known and is the main focus of this paper.

We start by studying the effects induced by the introduction of bundled discounts in each of the three scenarios identified above. Our main results are the following. First, and relative to the no-discounting benchmark case, the headline prices of the bundling firms rise (i.e. are inflated) whereas the headline prices of the firms not involved in discounting (if any) decrease. Second, whatever the scenario considered, bundled discounts always induce a decrease both in consumer surplus and in total welfare, with those reductions being more pronounced in the bilateral bundling case. This leads to the conclusion that in none of the three scenarios should bundled discounts be free of antitrust concerns. Third, in the case of bilateral bundling, both the price of the high quality bundle and the price of the low quality bundle, net of the corresponding discounts, rise. This last result is in sharp contrast with Gans and King (2006) since, in their model, the headline price for each separate good is increased by the exact value of the discount.

We then turn to the study of the firms' simultaneous decisions regarding their eventual participation in a bundled discount scheme. It turns out that offering a bundled discount is a dominant strategy both for the high quality firms and for the low quality firms. Hence, the Nash equilibrium of this game corresponds to the scenario of bilateral bundling, as in Gans and King (2006), where there are no consumers buying unpaired products: all consumers find it optimal to purchase a bundle and benefit from the corresponding discount. However, and in contrast with Matutes and Regibeau (1992), Gans and King (2006), Thanassoulis (2007) or Maruyama and Minamikawa (2009), in this bilateral bundling scenario all firms earn *higher profits* than in the *status quo* no-discounting situation and, therefore, firms do not find themselves in a prisoner's dilemma. In our setting, all firms have very strong incentives to participate in bilateral bundling. Nevertheless, this scenario is shown to be the one leading to the most adverse consequences both in terms of consumer welfare and in terms of social welfare. This then suggests that competition authorities should scrutinize in detail bundling discounting by independent producers of vertically differentiated goods.

The remainder of the paper is organized as follows. In Section 2, we lay down our general framework and specify the timing of the proposed game. In Section 3, we study the competitive effects of both unilateral bundling by low quality producers and by high quality producers as well as bilateral bundling, relative to the benchmark case where there is no bundling. Section 4 studies what we term as the discounting game, a simultaneous move game where firms make decisions about their eventual participation in a bundled discount scheme. Section 5 investigates the robustness of the main results obtained when the assumption that consumer valuations over the two products are uncorrelated is relaxed. In particular, this section considers two polar cases of perfectly correlated valuations. Finally, Section 6 concludes the paper.

2 The model

2.1 Firms

We consider the case of two distinct products, X, Y, each being sold by two firms, a high quality producer and a low quality producer. Denote by A_X, A_Y the two high quality producers and by B_X, B_Y the two low quality producers of products X and Y, respectively. There are no costs associated with the production of either product or quality level. We denote the price of the higher quality product by P_i and the price of the lower quality product by p_i, with $i = X, Y$.

Each pair of producers may agree to participate in a bundled discount scheme, where we assume that only firms of the same quality level may offer the bundled discounts together. Hence, we consider four different scenarios. Scenario 0 is the benchmark case in which there are no discounts. Scenarios 1 and 2 are unilateral bundling scenarios and refer, respectively, to the cases of a discount given by the low quality or by the high quality firms. Scenario 3 refers to the case of bilateral bundling: simultaneous bundled discounts offered by both the low and the high quality firms. Let γ_j, with $j = A, B$ denote the discount offered by the producers of j_X and j_Y. When $\gamma_j > 0$ we say that firms j_X and j_Y are partner firms in the discounting scheme.

In what follows, s_j represents an index of the quality of the product sold by firm j_i, with $j = A, B$ and $i = X, Y$. Let $s := s_A - s_B > 0$. We are interested in the profitability of relatively small discounts; thus we assume that no discount can be larger than the average market price before the introduction of the discount. We write the discount as $\gamma_j = \beta_j s$ with $j = A, B$. Moreover, α denotes the percentage of the discount financed by the producer of X.

2.2 Consumers

The way we model consumers' preferences for quality follows Gabszewicz and Thisse (1979) and Shaked and Sutton (1982). Consumers purchase at most one unit of each good. Consumers' net utility when purchasing product i from producer A_i is given by $V_i+\theta_i s_A-P_i$, whereas consumer's net utility when purchasing product i from producer B_i is given by $V_i+\theta_i s_B-p_i$, with $i=X,Y$. Throughout the paper we assume that V_i is sufficiently large so that the market is fully covered, i.e., every consumer purchases one unit of each good. Consumers do not get any extra benefit or any transaction costs reduction from purchasing goods of the same quality. Hence, in the absence of a discount, the demand for one product is independent of the demand for the other.

We assume that consumers' valuations for the quality of both products, (θ_X,θ_Y), are uniformly distributed in $[0,1]\times[0,1]$.[8] Let

$$
\begin{aligned}
\theta_i^{*a} &:= \frac{P_i-p_i-\gamma_A}{s},\\
\theta_i^{*b} &:= \frac{P_i-p_i+\gamma_B}{s},
\end{aligned}
$$

with $i=X,Y$.

The following lemma presents the relevant demand functions.

Lemma 1:

(i) Assume that $\left(\theta_Y^{*a},\theta_X^{*a},\theta_Y^{*b},\theta_X^{*b}\right)\in[0,1]^4$. *Then, the demand functions for each possible pair of products are given by:*

$$
\begin{aligned}
Q_{A_X,B_Y} &= \left(1-\frac{P_X-p_X+\gamma_B}{s}\right)\left(\frac{P_Y-p_Y-\gamma_A}{s}\right),\\
Q_{B_X,A_Y} &= \left(1-\frac{P_Y-p_Y+\gamma_B}{s}\right)\left(\frac{P_X-p_X-\gamma_A}{s}\right),\\
Q_{A_X,A_Y} &= \left(1-\frac{P_Y-p_Y-\gamma_A}{s}\right)\left(1-\frac{P_X-p_X-\gamma_A}{s}\right)-\frac{(\gamma_A+\gamma_B)^2}{2s^2},\\
Q_{B_X,B_Y} &= \left(\frac{P_Y-p_Y+\gamma_B}{s}\right)\left(\frac{P_X-p_X+\gamma_B}{s}\right)-\frac{(\gamma_A+\gamma_B)^2}{2s^2}.
\end{aligned}
$$

(ii) Assume that $\theta_Y^{*a}<0$, $\theta_X^{*a}<0$ *and* $\theta_Y^{*b}+\theta_X^{*a}\in[0,1]$. *Then, the demand functions for*

[8] In Section 5 it is shown that the main results derived under the assumption that consumer valuations over the two products are uncorrelated extend to the cases of perfectly correlated valuations.

each possible pair of products are given by:

$$\begin{aligned} Q_{A_X,A_Y} &= 1-\left(\frac{P_X-p_X+P_Y-p_Y+\gamma_B-\gamma_A}{s}\right)^2/2, \\ Q_{B_X,B_Y} &= \left(\frac{P_X-p_X+P_Y-p_Y+\gamma_B-\gamma_A}{s}\right)^2/2, \\ Q_{A_X,B_Y} &= Q_{B_X,A_Y}=0. \end{aligned}$$

■

Now, making use of Lemma 1, the demand function for each individual product can be obtained from:

$$\begin{aligned} Q_{A_X} &= Q_{A_X,B_Y}+Q_{A_X,A_Y} \text{ and } Q_{B_X}=Q_{B_X,A_Y}+Q_{B_X,B_Y} \\ Q_{A_Y} &= Q_{B_X,A_Y}+Q_{A_X,A_Y} \text{ and } Q_{B_Y}=Q_{A_X,B_Y}+Q_{B_X,B_Y}. \end{aligned}$$

Clearly, all quantities are a function of prices and discounts. For the sake of brevity, however, we write $Q_{A_X,B_Y}(P_X,p_X,P_Y,p_Y,\gamma_A,\gamma_B)$ as Q_{A_X,B_Y} and so forth. Also, when we say that a given consumer purchases, say, A_X, B_Y we mean that this consumer purchases the high quality version of good X and the low quality version of good Y.

2.3 Timing

The timing of the game played between firms is as follows:

1. Firms simultaneously agree to their bundled discount, if any.
2. Given the bundled discount(s), all firms set their prices simultaneously.
3. Given prices and discounts, customers decide from which producers to make their purchases.

Since the discount is set before the headline prices, we are assuming that it is easier for a firm to change its own price than to change the bundled discount. This is a natural assumption given that any firm is free to unilaterally change its price at a short notice, whereas changing the discount would involve a renegotiation with the partner firm.

3 Results

In this section, we investigate the consequences of bundled discounts relative to the situation where there is no bundling (and, thus, no discounts). More specifically, in what follows, we discuss, apart from the no-discounting equilibrium, three different scenarios: two involving unilateral bundling (either by the low quality producers of by the high quality producers) and a final one involving bilateral bundling.

3.1 Scenario 0: No discounting benchmark case

Consider first the benchmark case where $\gamma_A = \gamma_B = 0$. The corresponding demand functions are given by

$$Q_{A_i} = \frac{(s - P_i + p_i)}{s} \text{ and } Q_{B_i} = \frac{(P_i - p_i)}{s},$$

with $i = X, Y$. As expected, the demand for the high quality producer of good i depends only on the high and low quality prices of this specific good. In the absence of discounts, the decision to purchase a high or low quality version of one product is independent of the prices of the two different quality variants of the other product.

The equilibrium prices in this benchmark case are given by

$$P_X/s = P_Y/s = 2/3 \text{ and } p_X/s = p_Y/s = 1/3.$$

The corresponding quantities and profits are, respectively, equal to:

$$\begin{aligned} Q_{A_X,B_Y} &= Q_{B_X,A_Y} = 2/9; Q_{A_X,A_Y} = 4/9 \text{ and } Q_{B_X,B_Y} = 1/9 \\ \Pi_{A_X} &= \Pi_{A_Y} = 4s/9 \text{ and } \Pi_{B_X} = \Pi_{B_Y} = s/9 \end{aligned}$$

with $\theta_X^{*b} = \theta_Y^{*b} = \theta_X^{*a} = \theta_Y^{*a} = 1/3$. Consumers' purchasing choices in this no-discounting equilibrium are illustrated in Figure 1.

Note that the average price for product i, $i = X, Y$, is $\overline{p}_i = 5/9$. This is our assumed upper bound on the set of admissible discounts in the analysis of the three following scenarios.

Aggregate consumer surplus, CS_0, and social welfare, W_0, are given by

$$CS_0 = \left(11 - 2\frac{s_A}{s_B}\right)/9 \text{ and } W_0 = (8\frac{s_A}{s_B} + 1)/9.$$

It should be remarked at this point that, throughout the paper, we measure consumer surplus and welfare in units if s_B.

3.2 Scenario 1: Unilateral bundling by the low quality firms

In this section, we consider the effects of bundled discounts by the pair of low quality producers, assuming that the pair of high quality producers does not offer a discount. Hence, $\gamma_B > 0$ while $\gamma_A = 0$. The resulting division of consumers is as in Figure 2.

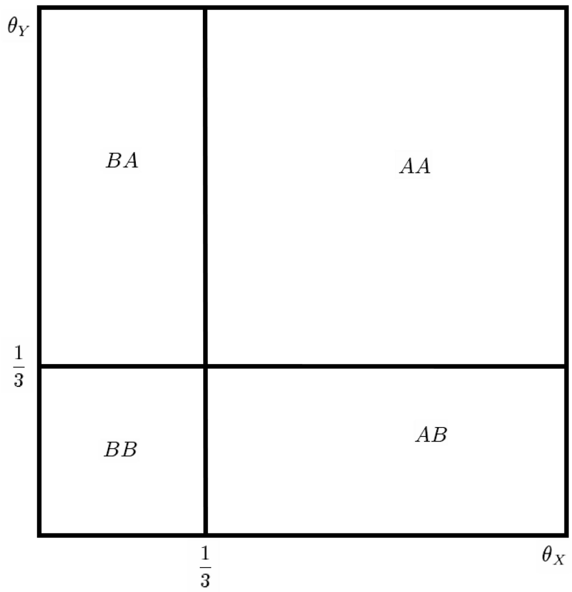

Figure 1: Consumers' choices with no discount.

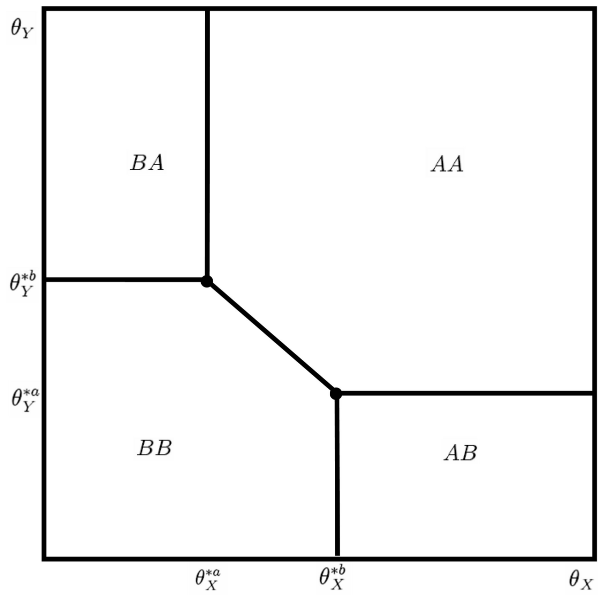

Figure 2: Consumers' choices with a unilateral bundled discount.

When this is the case, equilibrium prices result from the individual maximization of following objective functions:

$$\begin{aligned}\Pi_{A_X} &= P_X\left(Q_{A_X,B_Y}+Q_{A_X,A_Y}\right)\\ \Pi_{A_Y} &= P_Y\left(Q_{B_X,A_Y}+Q_{A_X,A_Y}\right)\\ \Pi_{B_X} &= p_X\left(Q_{B_X,A_Y}+Q_{B_X,B_Y}\right)-\gamma_B\alpha Q_{B_X,B_Y}\\ \Pi_{B_Y} &= p_Y\left(Q_{A_X,B_Y}+Q_{B_X,B_Y}\right)-\gamma_B\left(1-\alpha\right)Q_{B_X,B_Y}\end{aligned}$$

yielding

$$\begin{aligned}P_X(\alpha,\gamma_B) &= \frac{12s^4-\gamma_B^4\alpha\left(3-\alpha\right)+2\gamma_B^3 s\left(1-\alpha\right)\left(2-\alpha\right)+\gamma_B^2 s^2\left(2\alpha+2\alpha^2-9\right)-2\gamma_B s^3\left(1-\alpha\right)}{2s\left(9s^2-6\gamma_B^2-\alpha\gamma_B^2+\alpha^2\gamma_B^2\right)},\\ p_X(\alpha,\gamma_B) &= \frac{6s^4-\gamma_B^4\alpha\left(3-\alpha\right)-2\gamma_B^3 s\left(2\alpha+1\right)\left(2-\alpha\right)+\gamma_B^2 s^2\left(8\alpha+2\alpha^2-3\right)+2\gamma_B s^3\left(2\alpha+1\right)}{2s\left(9s^2-6\gamma_B^2-\alpha\gamma_B^2+\alpha^2\gamma_B^2\right)}.\end{aligned}$$

The remaining equilibrium prices can be obtained by making use of the fact that $P_Y(\alpha,\gamma_B) = P_X(1-\alpha,\gamma_B)$ and $p_Y(\alpha,\gamma_B) = p_X(1-\alpha,\gamma_B)$.

By definition, the joint profit of the low quality producers is given by $\Pi_B(\alpha) := \Pi_{B_X} + \Pi_{B_Y}$. Now, maximizing $\Pi_B(\alpha)$, evaluated at the equilibrium prices specified above, with respect to α, gives the relevant solution at $\alpha = \frac{1}{2}$.[9] Regardless of the level of the discount, firms optimally agree to fund it equally, which is a natural consequence of the symmetry assumptions in the model. This then implies that equilibrium prices boil down to:[10]

$$P_X = P_Y = \frac{\left(6\beta_B+\beta_B^3+8\right)s}{2\left(5\beta_B+6\right)} \text{ and } p_X = p_Y = \frac{\left(6\beta_B+6\beta_B^2+\beta_B^3+4\right)s}{2\left(5\beta_B+6\right)}.$$

Analyzing the equilibrium prices, we conclude that, for all admissible discounts:

(i) the low quality headline price increases with β_B. This is as expected. All else constant, a higher discount offered by the low quality producers increases the demand for the bundle, which means that demand for each low quality component increases, thus leading to higher headline prices for these component goods. Also, the introduction of the discount can be interpreted as a unit cost, partially incurred by each partner firm, for the units entitled to the discount.

[9] This is the unique solution to $\partial\Pi_B(\alpha)/\partial\alpha = 0$ for any $\gamma_B \in [0, 5s/9]$. Second-order conditions are always verified in this range.

[10] In order to compare Figures 1 and 2 note that, in equilibrium, $\theta_X^{*a} = P_X/s - p_X/s \le \frac{1}{3}$ and that $\theta_X^{*b} = P_X/s - p_X/s + \beta_B \ge \frac{1}{3}$. Furthermore, $\frac{1}{3} - \theta_X^{*a} < \theta_X^{*b} - \frac{1}{3}$. Ehen β_B is equal to 0, we obtain the standard benchmark solution wherein $P_X = P_Y = 2s/3$ and $p_X = p_Y = s/3$.

(*ii*) the low quality bundle price, net of the discount, is decreasing in β_B. Despite the fact that both headline low quality prices increase with the discount, their sum increases at a lower rate than the discount itself and, as a result, the "net" bundle price decreases with the discount.

(*iii*) the high quality headline price is a U-shaped function of β_B. This results from the interaction of two effects with opposite signs. On the one hand, the price of the low quality bundle is decreasing in β_B, which will make some consumers switch from A_X, B_Y or B_X, A_Y to B_X, B_Y, thereby reducing the demand for the high quality component. However, the increase in the headline price of the low quality component, will also make some consumers switch from A_X, B_Y or B_X, A_Y to A_X, A_Y, thus increasing the demand for the high quality component. Each effect may then dominate the other.

(*iv*) the sum of the headlines prices of two different quality products increases with the discount. When those headline prices increase with the discount, this effect is obvious. When, however, the high quality product headline price decreases in β_B whereas the low quality product headline price increases, it turns out that the latter effect is stronger.

The corresponding equilibrium quantities are equal to:

$$
\begin{aligned}
Q_{A_X,B_Y} &= Q_{B_X,A_Y} = \frac{\left(\beta_B + 2\beta_B^2 - 4\right)\left(3\beta_B^2 - 2\right)}{\left(5\beta_B + 6\right)^2}, \\
Q_{A_X,A_Y} &= \frac{\left(7\beta_B^4 - 62\beta_B^2 - 80\beta_B - 32\right)}{-2\left(5\beta_B + 6\right)^2}, \\
Q_{B_X,B_Y} &= \frac{\left(12\beta_B^3 - 52\beta_B^2 - 48\beta_B + 17\beta_B^4 - 8\right)}{-2\left(5\beta_B + 6\right)^2}.
\end{aligned}
$$

Two remarks are worth making regarding the equilibrium quantities, which follow directly from the price effects described above. First, both Q_{A_X,A_Y} and Q_{B_X,B_Y} increase with the level of the discount given and are larger than the corresponding quantities in the absence of discounting. Second, Q_{A_X,B_Y} (Q_{B_X,A_Y}) decreases with β_B: the share of consumers opting for purchasing products of different quality decreases with the discount level.

The optimal discount is obtained by maximizing the low quality firms' joint profit evaluated at the equilibrium prices,

$$
\Pi_B\left(\frac{1}{2}\right) = \frac{32\beta_B - 28\beta_B^3 + 10\beta_B^4 + 11\beta_B^5 - \beta_B^6 + 16}{2\left(5\beta_B + 6\right)^2} s,
$$

with respect to β_B and subject to $\beta_B \in \left[0, \frac{5}{9}\right]$. This yields $\beta_B^* = 0.14058$ (obtained numerically).[11]

This optimal discount results from the trade-off between several effects, which we discuss in turn. The discount impacts aggregate profit directly but also strategically, via the equilibrium prices. The *direct effect* is given by:

$$\frac{\partial \Pi_B}{\partial \gamma_B} = \frac{\partial \Pi_B}{\partial Q_{B_X,A_Y}} \frac{\partial Q_{B_X,A_Y}}{\partial \gamma_B} + \frac{\partial \Pi_B}{\partial Q_{A_X,B_Y}} \frac{\partial Q_{A_X,B_Y}}{\partial \gamma_B} + \frac{\partial \Pi_B}{\partial Q_{B_X,B_Y}} \frac{\partial Q_{B_X,B_Y}}{\partial \gamma_B} + \frac{\partial \Pi_B}{\partial \gamma_B}.$$

Evaluated at the no-discount equilibrium, the direct effect boils down to:

$$\frac{s}{3}\left(\frac{\frac{s}{3}}{-s^2}\right) + \frac{s}{3}\left(\frac{\frac{s}{3}}{-s^2}\right) + \frac{2s}{3}\frac{\left(\frac{2s}{3}\right)}{s^2} - \frac{1}{9} = \frac{1}{9}.$$

This effect works as follows. When the discount increases, some consumers purchasing B_X, A_Y and A_X, B_Y, as well as A_X, A_Y, will switch to the bundle B_X, B_Y. Hence, the demand for the low quality firms will increase. However, all those consumers that were previously purchasing B_X, B_Y will now benefit from the increased discount.[12] The net effect is positive.

Let us now discuss the effect on the partner firms' profits that is due to the change in the high quality prices in response to an increase in the discount. As this results from changes in the prices of the non-partner firms, we call this the *external strategic effect*:

$$\begin{aligned} & \sum_{i=X,Y} \frac{\partial \left(\Pi_{B_X} + \Pi_{B_Y}\right)}{\partial P_i} \frac{\partial P_i}{\partial \gamma_B} \\ = & \sum_{i=X,Y} \left(p_X \frac{\partial \left(Q_{B_X,A_Y} + Q_{B_X,B_Y}\right)}{\partial P_i} + p_Y \frac{\partial \left(Q_{A_X,B_Y} + Q_{B_X,B_Y}\right)}{\partial P_i} - \gamma_B \frac{\partial Q_{B_X,B_Y}}{\partial P_i} \right) \frac{\partial P_i}{\partial \gamma_B}. \end{aligned}$$

Evaluated at the no-discount equilibrium, this external strategic effect boils down to:

$$\left(s^{-2}\left(s\right)\left(\frac{s}{3} + \frac{s}{3}\right) - s^{-2}\left(\frac{2s}{3}\right) * 0 \right) \frac{(-2)}{(6)^2} = -\frac{1}{27}$$

An increase in the discount will induce a decrease in the headline prices for the high quality products. As a result, some consumers will switch from B_X, B_Y, B_X, A_Y and A_X, B_Y to A_X, A_Y, as well others will switch from B_X, B_Y to B_X, A_Y and A_X, B_Y. This will result in a decrease in the quantity demanded of the low quality firms but also in a reduction of customers entitled to the discount. This effect is negative and mitigates the previous one.

[11]For an interior solution, all $\theta_X^{*b}, \theta_Y^{*b}, \theta_X^*, \theta_Y^*$ must belong to the $[0, 1]$ interval. This is true if and only if $\beta_B < \sqrt{6}/3$ which holds in the feasible range of discounts.

[12]The direct effects on demand are illustrated in Figure 3.

Finally, we present the (cross) effect in the partner firms' profits that is due to the change in the low quality products' prices in response to an increase in the discount. We call this the *internal strategic effect*:

$$\begin{aligned}
&\frac{\partial \Pi_{B_Y}}{\partial p_X}\frac{\partial p_X}{\partial \gamma_B}+\frac{\partial \Pi_{B_X}}{\partial p_Y}\frac{\partial p_Y}{\partial \gamma_B} \\
=\ &\left(p_Y\left(\frac{\partial Q_{A_X,B_Y}}{\partial p_X}+\frac{\partial Q_{B_X,B_Y}}{\partial p_X}\right)-\frac{\gamma_B}{2}\frac{\partial Q_{B_X,B_Y}}{\partial p_X}\right)\frac{\partial p_X}{\partial \gamma_B}+ \\
&+\left(p_X\left(\frac{\partial Q_{B_X,A_Y}}{\partial p_Y}+\frac{\partial Q_{B_X,B_Y}}{\partial p_Y}\right)-\frac{\gamma_B}{2}\frac{\partial Q_{B_X,B_Y}}{\partial p_Y}\right)\frac{\partial p_Y}{\partial \gamma_B}.
\end{aligned}$$

Evaluated at the no-discount equilibrium, this internal strategic effect boils down to

$$\left(\frac{s}{3}\left(\frac{\frac{s}{3}}{s^2}-\frac{\frac{s}{3}}{s^2}\right)+\frac{s}{3}\left(\frac{\frac{s}{3}}{s^2}-\frac{\frac{s}{3}}{s^2}\right)\right)\frac{8}{36}=0$$

An increase in the discount will lead to an increase in the headline prices for the low quality products. The low quality producer of product Y, firm B_Y, will be affected by the increase in the headline price of the low quality version of product X in the following way. Some consumers of the B_X, B_Y will switch to A_X, B_Y, leaving the total demand of the low quality producer of Y unchanged.[13] There is also a positive effect which is linked to the reduction of customers entitled to the discount. However, at the no-discount equilibrium, this last effect does not change the profit of the low quality producer of Y. Likewise for the low quality producer of X.

Now, at the optimal discount level, i.e., when $\beta_B^*=0.14058$, we have that:[14]

$$\begin{aligned}
P_X &= P_Y=0.65988s \text{ and } p_X=p_Y=0.37035s, \\
Q_{A_X,B_Y} &= Q_{B_X,A_Y}=0.165; Q_{A_X,A_Y}=0.49488 \text{ and } Q_{B_X,B_Y}=0.17512.
\end{aligned}$$

Aggregate consumer surplus, CS_1, and social welfare, W_1, are given by

$$CS_1=\frac{6.7578-1.3193\frac{s_A}{s_B}}{5.4385} \text{ and } W_1=\frac{8.5678\frac{s_A}{s_B}+1.1658}{9.7335}.$$

The following proposition summarizes these results:

Proposition 1: *If only the low quality firms can offer the bundled discount, then, in equilibrium, and relative to the situation without bundling:*

(i) the headline prices for the low quality bundling firms will rise.

[13] Also, some consumers of B_X, A_Y will switch to A_X, A_Y but this does not affect the total demand for B_Y.

[14] The equilibrium configuration of demand can be observed in Figure 2, where $\theta_X^{*a}=\theta_Y^{*a}:=0.28953$ and $\theta_X^{*b}=\theta_Y^{*b}:=0.43011$.

(ii) the headline prices for the high quality firms will fall.

(iii) the price of the bundle, net of the discount, will fall.

(iv) both consumer surplus and welfare will fall. ■

It is straightforward to conclude that with unilateral bundling by the low quality producers, the average equilibrium price decreases from 1.1111 to 1.0982. It should be remarked, however, that, in our setting, price is only a good indicator to evaluate the discount effects on consumer welfare if one restricts attention to those consumers who keep the same purchase option after the discount introduction (for the other consumers, quality will change as well). So, the following table illustrates the prices paid before and after the introduction of the discount for those consumers whose purchase options were not affected by the discount.

Total price paid by product X and Y

	Buy B_X, B_Y	Buy A_X, A_Y	Buy B_X, A_Y or A_X, B_Y
Without discount	0.6666	1.3333	1.0000
With discount	0.6001	1.3198	1.0302
% increase in price	−9.98%	−1.01%	+3.02%
% consumers	11.11%	44.30%	33%

Consumers' who either purchase the low quality bundle or the two high quality products will pay a lower total price while the remainder will pay a higher total price. Additionally, some consumers will make different options after the introduction of the bundled discount and, thus, will face a price/quality trade-off. Consumer surplus takes all the induced changes in prices and quality into account as well as consumer heterogeneity. As the previous proposition shows, however, it turns out that, overall, consumers will be worse off with the discount introduction.

Simple algebra shows that equilibrium profits are given by:

$$\Pi_{A_X} = \Pi_{A_Y} = 0.435\,44s \text{ and } \Pi_{B_X} = \Pi_{B_Y} = 0.113\,65s.$$

Hence, offering a bundled discount to those consumers opting for the acquisition of the low quality bundle will increase the individual profit of the firms offering the discount and, at the same time, will decrease the profit of the high quality producers (not involved in a similar discounting scheme). In addition, aggregate industry profit will fall and, given that consumer surplus will also fall, total welfare will be reduced.

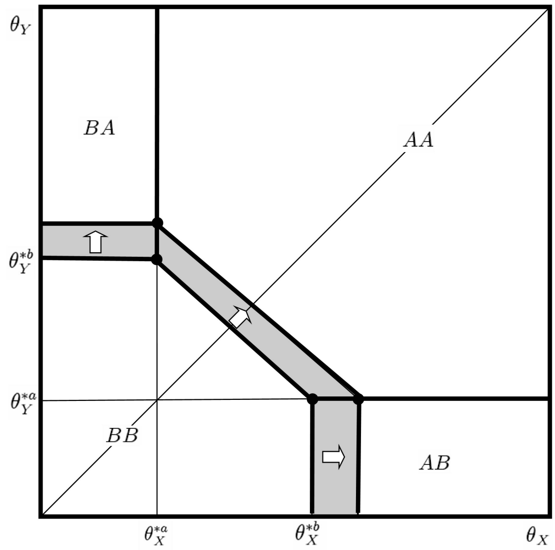

Figure 3: Direct effect on demand of an increase in the bundled discount by the low quality producers.

3.3 Scenario 2: Unilateral bundling by the high quality firms

We now analyze the scenario in which $\gamma_A > 0$ while $\gamma_B = 0$. In this scenario, equilibrium prices result from the maximization of following objective functions,

$$\begin{aligned}
\Pi_{A_X} &= P_X\left(Q_{A_X,B_Y} + Q_{A_X,A_Y}\right) - \gamma_A \alpha Q_{A_X,A_Y} \\
\Pi_{A_Y} &= P_Y\left(Q_{B_X,A_Y} + Q_{A_X,A_Y}\right) - \gamma_A\left(1-\alpha\right) Q_{A_X,A_Y} \\
\Pi_{B_X} &= p_X\left(Q_{B_X,A_Y} + Q_{B_X,B_Y}\right) \\
\Pi_{B_Y} &= p_Y\left(Q_{A_X,B_Y} + Q_{B_X,B_Y}\right)
\end{aligned}$$

yielding

$$\begin{aligned}
P_X(\alpha,\gamma_A) &= \frac{\left(6s^3 - 4s\gamma_A^2 - s^2\gamma_A + \gamma_A\alpha\left(4s^2 - 3\gamma_A^2\right) + \gamma_A^2\alpha^2\left(2s+\gamma_A\right)\right)\left(2s+\gamma_A\right)}{2\left(9s^2 - 6\gamma_A^2 - \alpha\gamma_A^2 + \alpha^2\gamma_A^2\right)s}, \\
p_X(\alpha,\gamma_A) &= \frac{s\left(6s^3 + 4\gamma_A^3 - 3s\gamma_A^2 - 4s^2\gamma_A\right) + \gamma_A\alpha\left(4s^3 - 3\gamma_A^3 - 6s\gamma_A^2\right) + \gamma_A^2\alpha^2\left(2s\gamma_A + 2s^2 + \gamma_A^2\right)}{2\left(9s^2 - 6\gamma_A^2 - \alpha\gamma_A^2 + \alpha^2\gamma_A^2\right)s},
\end{aligned}$$

where, as before, the remaining two equilibrium prices can be easily obtained from $P_Y(\alpha,\gamma_A) = P_X(1-\alpha,\gamma_A)$ and $p_Y(\alpha,\gamma_A) = p_X(1-\alpha,\gamma_A)$.

By definition, the joint profits of the high quality firms are $\Pi_A(\alpha) := \Pi_{A_X} + \Pi_{A_Y}$. By maximizing $\Pi_A(\alpha)$, evaluated at the equilibrium prices above, with respect to α, one obtains the relevant solution at $\alpha = \frac{1}{2}$.[15]

[15] This is the unique solution to $\partial\Pi_A(\alpha)/\partial\alpha = 0$ for any $\gamma_A \in \left[0, \frac{5}{9}s\right]$. Second-order conditions are always verified in this range.

Now, since $\gamma_A = \beta_A s$ and $\alpha = \frac{1}{2}$, equilibrium prices boil down to:[16]

$$P_X = P_Y = \frac{(\beta_A + 2)^3 s}{2(5\beta_A + 6)} \text{ and } p_X = p_Y = \frac{\left(2\beta_A + \beta_A^3 + 4\right) s}{2(5\beta_A + 6)}.$$

Analyzing the equilibrium prices, we conclude that, for all admissible discounts:

(*i*) the high quality headline price increases with β_A. This occurs for the same reason that the low quality headline price increased with β_B in the previous subsection.

(*ii*) the high quality bundle price, net of the corresponding discount, is U-shaped (convex) in β_A, which contrasts with the case of unilateral bundling by the low quality firms. Comparison of Figures 3 and 4 shows that the effect of an increase in the discount on a given firm's demand is larger in the case of a discount by the high quality producers. Hence, the headline price of a high quality product increases more and may compensate the increase in the discount.

(*iii*) the low quality headline price decreases in β_A. This may be counterintuitive, especially when the high quality headline price and the high quality bundle's "net" price are both increasing with the discount level. The intuition is, however, simple. When both of these prices are increasing, the headline high quality price increases at a higher rate than the bundle "net" price. Hence, some consumers that were previously purchasing a mixed quality pair (i.e., A_X, B_Y or B_X, A_Y) will switch to the high quality bundle while others will switch to the low quality pair. The net effect on the demand for the low quality firms may thus be negative when the consumers switching from A_X, B_Y or B_X, A_Y to A_X, A_Y outnumber those switching to B_X, B_Y. Direct inspection of Figure 1 suggests that, indeed, for most consumers initially buying A_X, B_Y or B_X, A_Y, the consumers purchasing the A_X, A_Y bundle are "closer" in terms of preferences for quality than those purchasing the B_X, B_Y pair. The same reasoning applies to the case when the price of the high quality bundle is decreasing, wherein the switching from A_X, B_Y or B_X, A_Y to A_X, A_Y is reinforced.

(*iv*) The sum of the headlines prices of two goods of different qualities increases with β_A.

[16] As in scenario 1, we have that, in equilibrium, $\theta_X^{*b} = P_X/s - p_X/s \geq \frac{1}{3}$ and that $\theta_X^{*a} = P_X/s - p_X/s - \beta_A \leq \frac{1}{3}$. Furthermore, $\theta_X^{*b} - \frac{1}{3} > \frac{1}{3} - \theta_X^{*a}$. Hence, Figure 2 also applies qualitatively to this case.

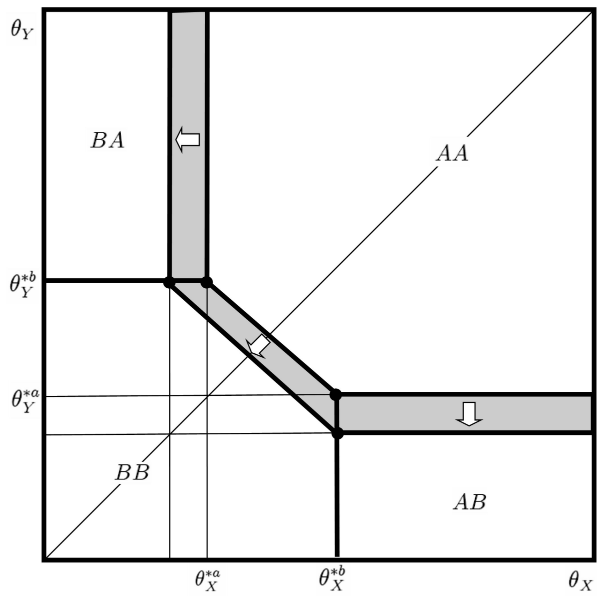

Figure 4: The direct effect on demand of increasing the bundled discount by the high quality producers.

The corresponding equilibrium quantities are given by:

$$
\begin{aligned}
Q_{A_X,B_Y} &= Q_{B_X,A_Y} = \frac{\left(3\beta_A^2-4\right)\left(\beta_A+2\beta_A^2-2\right)}{\left(5\beta_A+6\right)^2} \\
Q_{A_X,A_Y} &= \frac{\left(12\beta_A^3-68\beta_A^2-96\beta_A+17\beta_A^4-32\right)}{-2\left(5\beta_A+6\right)^2} \\
Q_{B_X,B_Y} &= \frac{\left(7\beta_A^4-38\beta_A^2-40\beta_A-8\right)}{-2\left(5\beta_A+6\right)^2}
\end{aligned}
$$

Again, two remarks are in order regarding these equilibrium quantities, which follow directly from the price effects described above. First, both Q_{A_X,A_Y} and Q_{B_X,B_Y} increase with the level of the discount given by the high quality producers and are larger than the corresponding quantities in the absence of discounting. Second, Q_{A_X,B_Y} (Q_{B_X,A_Y}) decreases with β_A: the share of consumers opting for purchasing products of different quality decreases with the discount level.

Now, the optimal discount is obtained by maximizing joint profit,

$$
\Pi_A(\frac{1}{2}) = \frac{\left(16\beta_A^2-64\beta_A+4\beta_A^3-12\beta_A^4+\beta_A^5-64\right)\left(\beta_A+1\right)}{-2\left(5\beta_A+6\right)^2}s,
$$

with respect to β_A.

For an interior solution to exist, all $\theta_X^{*b}, \theta_Y^{*b}, \theta_X^{*}, \theta_Y^{*}$ must belong to the $[0,1]$ interval. This is true for all $\beta_A \in \left[0, \frac{5}{9}\right]$. So, maximizing $\Pi_A(1/2)$ with respect to β_A subject to $\beta_A \in \left[0, \frac{5}{9}\right]$ yields $\beta_A^* = \frac{5}{9}$, a corner solution.

The trade-off faced by the partner high-quality producers when deciding on the optimal level for the discount is qualitatively similar to the one discussed in the previous section for the low quality partner firms. There are, however, the following differences when the effects are evaluated at the no-discount equilibrium:

- The initial number of consumers entitled to the discount is four times larger;
- Each consumer that switches to the high quality components is charged a price which is twice as large as the price paid by consumers who switch to low quality components. The unit cost, however, is the same for both types of quality, and equal to 0.
- The effects of the discount on the corresponding quantities are larger (again, twice as large) - see Figures 3 and 4.
- The effect of the discount on the equilibrium headline prices of the bundling partners (positive effect) and of the non bundling partners (negative effect) is larger in the case of bundling by the high quality firms (again, twice as large).

As a result, the overall effect (and also the direct and the strategic effects when taken separately) turns out to be four times larger in the scenario under analysis than in the case of the bundle by the low quality firms. This helps explaining why the magnitude of the equilibrium discount is larger in the case of the high quality bundle.

At the optimal discount level, i.e. when $\beta_A^* = \frac{5}{9}$, we have that:[17]

$$\begin{aligned} P_X &= P_Y = \frac{12\,167}{12\,798}s \text{ and } p_X = p_Y = \frac{3851}{12\,798}s, \\ Q_{A_X,B_Y} &= Q_{B_X,A_Y} = \frac{67 \times 83}{3^3 79^2}; Q_{A_X,A_Y} = \frac{673\,447}{3^4 79^2 2} \text{ and } Q_{B_X,B_Y} = \frac{439 \times 617}{3^4 79^2 2}. \end{aligned}$$

These results are summarized in the following proposition.

Proposition 2: *If only the high quality firms can offer a bundled discount, then, in equilibrium and relative to the situation without bundling:*

(i) the headline prices for the high quality bundling firms will rise.

(ii) the headline prices for the low quality firms will fall.

(iii) the price of the bundle, net of the discount, will rise.

(iv) both consumer surplus and welfare will fall. ■

[17] The equilibrium configuration of demand can be observed in Figure 2, where $\theta_X^{*a} = \theta_Y^{*a} := 67/711$ and $\theta_X^{*b} = \theta_Y^{*b} := 154/237$.

Contrary to what happened for the case discussed in the previous section, in the current scenario, the introduction of a bundled discount (by the high quality producers) leads to an increase in average price from 1.1111 to 1.140 3. In addition, the prices paid before and after the introduction of the discount by those consumers who make the same purchasing options after the discount becomes available are summarized in the following table.

Total price paid by product X and Y			
	Buy B_X, B_Y	Buy A_X, A_Y	Buy B_X, A_Y or A_X, B_Y
Without discount	0.6666	1.3333	1.0000
With discount	0.6018	1.3458	1.2516
% increase in price	−9.72%	0.94%	+25.16%
% consumers	11.11%	44.15%	6.60%

Hence, within the group of consumers keeping their purchase decision, the only consumers who end up benefiting from the discount introduction are those who *do not* purchase any product from the bundling firms. In addition, and in line with the scenario studied in the previous section, overall, consumer surplus will be negatively affected by the introduction of a bundled discount by the high quality producers.

Some algebra shows that the equilibrium profits are given by:

$$\begin{aligned}\Pi_{A_X} &= \Pi_{A_Y} = \frac{39\,276\,517}{81\,894\,402}s = 0.479\,60s,\\ \Pi_{B_X} &= \Pi_{B_Y} = \frac{14\,830\,201}{163\,788\,804}s = 9.054\,5\times 10^{-2}s.\end{aligned}$$

Thus, the bundling firms will benefit from an increase in profits whereby their competitors will see profits declining. In addition, and despite the fact that aggregate profits increase, total welfare decreases upon the introduction of the discount.

3.4 Scenario 3: Bilateral bundling

We now analyze the case in which γ_A and γ_B are both positive. We assume from the outset that $\alpha_A = \alpha_B = \frac{1}{2}$, where α_i denotes the percentage of the discount financed by the producer of product X with quality $i = A, B$.

If prices are such that there are *four distinct groups of consumers* (two groups buying only products of the same quality, A_X, A_Y or B_X, B_Y, and two other groups buying products of different quality, A_X, B_Y or B_X, A_Y), then firms maximize the following objective functions:

$$\begin{aligned}
\Pi_{A_X} &= P_X \left(Q_{A_X,B_Y} + Q_{A_X,A_Y}\right) - \gamma_A \alpha_A Q_{A_X,A_Y} \\
\Pi_{A_Y} &= P_Y \left(Q_{B_X,A_Y} + Q_{A_X,A_Y}\right) - \gamma_A \left(1-\alpha_A\right) Q_{A_X,A_Y} \\
\Pi_{B_X} &= p_X \left(Q_{B_X,A_Y} + Q_{B_X,B_Y}\right) - \gamma_B \alpha_B Q_{B_X,B_Y} \\
\Pi_{B_Y} &= p_Y \left(Q_{A_X,B_Y} + Q_{B_X,B_Y}\right) - \gamma_B \left(1-\alpha_B\right) Q_{B_X,B_Y}
\end{aligned}$$

yielding

$$\begin{aligned}
P_X &= P_Y = \frac{\left(12\beta_A + 6\beta_B + 5\beta_A\beta_B + 6\beta_A^2 + \beta_A^3 + \beta_B^3 + 4\beta_A\beta_B^2 + 4\beta_A^2\beta_B + 8\right)}{2\left(5\beta_A + 5\beta_B + 6\right)} s \\
p_X &= p_Y = \frac{\left(2\beta_A + 6\beta_B + 5\beta_A\beta_B + \beta_A^3 + 6\beta_B^2 + \beta_B^3 + 4\beta_A\beta_B^2 + 4\beta_A^2\beta_B + 4\right)}{2\left(5\beta_A + 5\beta_B + 6\right)} s
\end{aligned}$$

where, as before, we have written $\gamma_A = \beta_A s$ and $\gamma_B = \beta_B s$. Aggregate profits are easily obtained by substituting the equilibrium prices and quantities in $\Pi_A = \Pi_{A_X} + \Pi_{A_Y}$ and $\Pi_B = \Pi_{B_X} + \Pi_{B_Y}$. As $\frac{\partial^2 \Pi_A}{\partial\beta_A \partial\beta_B} > 0$, the high quality firms will choose $\beta_A^* = \frac{5}{9}$, for any $\beta_B \geq 0$, as long as θ_X^{*b},θ_Y^{*b},θ_X^{*a},θ_Y^{*a} are all in the $[0,1]$ interval. It should also be noted that $\theta_X^{*b} := \theta_Y^{*b} := \left(5\beta_A + 6\beta_B + 5\beta_A\beta_B + 3\beta_A^2 + 2\beta_B^2 + 2\right) / \left(5\beta_A + 5\beta_B + 6\right)$ is always in $[0,1]$ and that $\theta_X^{*a} = \theta_Y^{*a} := \left(2 - 5\beta_A\beta_B - 2\beta_A^2 - 3\beta_B^2 - \beta_A\right) / \left(5\beta_A + 5\beta_B + 6\right)$ is always lower than 1. However, whenever $\beta_B > \frac{1}{6}\sqrt{\beta_A^2 - 12\beta_A + 24} - \frac{5}{6}\beta_A$, θ_X^{*a} is negative. In particular, if $\beta_A = \frac{5}{9}$, θ_X^{*a} is negative for $\beta_B > 0.237\,08$. Hence, for any $\beta_B \in [0, 0.237\,08]$,

$$\Pi_B|_{\beta_A=\frac{5}{9}} = \frac{9\beta_B\left(9\beta_B\left(81\beta_B\left(\beta_B\left(9\beta_B\left(9\beta_B - 74\right) - 2840\right) - 1007\right) + 25\,730\right) - 2029\,393\right) - 14\,830\,201}{-13\,122\left(45\beta_B + 79\right)^2} s$$

is maximized at $\beta_B = 0.128\,92$, with $\Pi_B|_{\beta_A=\frac{5}{9},\beta_B=0.128\,92} = 0.18195$. However, for values of $\beta_B \in \left(0.237\,08, \frac{5}{9}\right]$ we now show that the low quality firms may obtain a higher profit than 0.18195.

If $\beta_A = \frac{5}{9}$, any $\beta_B \in \left(0.237\,08, \frac{5}{9}\right]$ will result in $\theta_X^{*a} < 0$. This being the case, there will be only *two groups of consumers*: consumers purchasing the high quality pair of products, A_X, A_Y, and consumers purchasing the low quality pair of products, ,$B_X B_Y$. The corresponding demand functions are as given by case (ii) in Lemma 1. At a symmetric equilibrium, we obtain:

$$\begin{aligned}
P_X &= P_Y = \left(\frac{1}{2}\beta_A + \frac{1}{3}\sqrt{6}\right) s \text{ and } p_X = p_Y = \left(\frac{1}{2}\beta_B + \frac{1}{6}\sqrt{6}\right) s \\
Q_{A_X,A_Y} &= \frac{2}{3}; Q_{B_X,B_Y} = \frac{1}{3} \text{ and } Q_{A_X,B_Y} = Q_{B_X,A_Y} = 0,
\end{aligned}$$

which verifies the second order conditions.[18]

Analyzing these equilibrium prices, we conclude that, for all admissible discounts:

[18]Note that $\theta_Y^{*b} + \theta_X^{*a} = \frac{\sqrt{6}}{3} \in [0,1]$.

(i) the high quality and the low quality headline prices increase with the respective discount.

(ii) the high and low quality bundle prices are *independent* of the discount, with $p_X + p_Y - \gamma_B = \frac{\sqrt{6}}{3}s$ and $P_X + P_Y - \gamma_A = \frac{2\sqrt{6}}{3}s$.

(iii) the sum of the headlines prices of the two different quality products increases with the discounts.

Simple algebra shows that equilibrium profits are then:

$$\Pi_{A_X} = \Pi_{A_Y} = \frac{2\sqrt{6}}{9}s = 0.544\,33s \text{ and } \Pi_{B_X} = \Pi_{B_Y} = \frac{\sqrt{6}}{18}s = 0.136\,08s,$$

which are independent of both discount levels. Moreover, the (equilibrium) profit for the low quality firms exceeds the one they would obtain by setting $\beta_B = 0.128\,92$.

Proposition 3: *If both the low and high quality firms can offer the bundled discount, then, in equilibrium and relative to the situation without bundling:*

(i) the headline prices for the high quality bundling firms will rise.

(ii) the headline prices for the low quality firms will rise.

(iii) the price of the high quality bundle, net of the discount, will rise .

(iv) the price of the low quality bundle, net of the discount, will rise.

(iv) both consumer surplus and welfare will fall. ■

Figure 5 illustrates the outcome under bilateral bundling. Interestingly, the competition for customers drives the discount levels up to the point that every consumer decides to buy one of the available bundles. After the introduction of the two discounts, the prices of the separate products of different quality become unattractive and, as a result, no consumer decides, in equilibrium, to mix-and-match, purchasing products of different quality.

With bilateral bundled discounts, average price increases from 1.1111 to 1.3608. The following table summarizes the prices paid by those consumers who keep the same purchasing decisions after the introduction of both discounts.

Total price paid by product X and Y			
	Buy B_X, B_Y	Buy A_X, A_Y	Buy B_X, A_Y or A_X, B_Y
Without discount	0.6666	1.3333	1.0000
With discount	0.8165	1.6330	–
% increase in price	+22.49%	+22.48%	–
% consumers	11.11%	43.32%	–

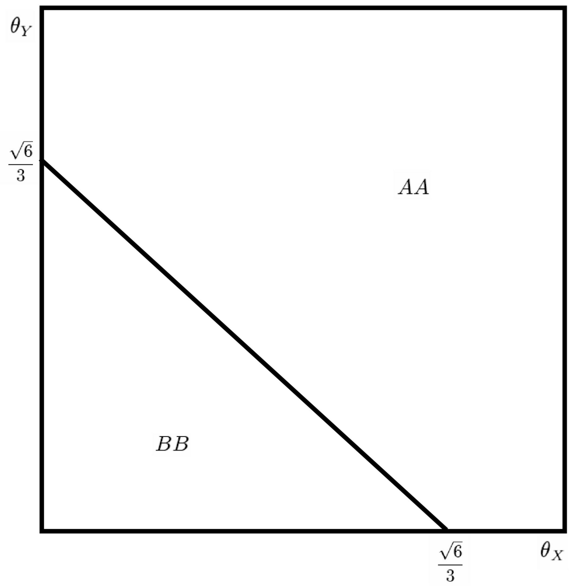

Figure 5: Consumers' choices with bilateral bundled discounts.

Note that consumers who formerly purchased goods of the same quality will be worse off with the introduction of the two bundled discounts. In addition, in this bilateral bundling scenario, both pairs of firms will be better off. However, total welfare will decrease.

4 The Discount Game

In this section, we discuss the simultaneous decisions by the high quality and low quality firms on whether to participate in a bundled discount scheme. The payoff matrix below follows directly from the previous results.[19]

		Low quality firms	
		No discount	Discount
High quality firms	No discount	$444.44; 111.11$	$435.44; 113.65$
	Discount	$479.60; 90.5$	$544.33; 136.08$

The following table presents the absolute and relative increases in profit in each of the three scenarios considered. In case of unilateral bundling, the high quality pair has a higher incentive, both in absolute or in relative terms, to offer the bundled discount. Curiously, however, in case of bilateral bundling, the relative increase in profit is almost the same for the high and low quality firms.

[19] All payoffs were multiplied by $1000/s$.

	Low quality firms		High quality firms	
	$\Delta\Pi_{B_i}$	$\Delta\Pi_{B_i}/\Pi_{B_i}$	$\Delta\Pi_{A_i}$	$\Delta\Pi_{A_i}/\Pi_{A_i}$
Scenario 1	$+2.54$	$+2.29\%$	-9.00	-2.03%
Scenario 2	-20.10	-18.55%	$+35.16$	$+7.91\%$
Scenario 3	$+24.97$	$+22.47\%$	$+99.89$	$+22.48\%$

Clearly, offering a bundled discount is a dominant strategy both for the high quality firms and for the low quality firms. Hence, the Nash equilibrium of this game corresponds to the scenario of bilateral bundling, as in Gans and King (2006), where there are no consumers buying unpaired products: all consumers find it optimal to purchase a bundle and benefit from the corresponding discount. However, and in contrast with Matutes and Regibeau (1992), Gans and King (2006), Thanassoulis (2007) or Maruyama and Minamikawa (2009), in this bilateral bundling scenario all firms earn *higher profits* than in the *status quo* no-discounting situation. So, in our setting, firms do not find themselves in the prisoners' dilemma: all firms have very strong incentives to participate in bilateral bundling. Nevertheless, and as shown above, this scenario is the one leading to the most adverse consequences both in terms of consumer welfare and in terms of social welfare. This then suggests that competition authorities should be more vigilant with regards to bundling discounting by independent producers of vertically differentiated goods.

5 Extensions

So far, we have assumed that consumers' valuation for quality is uncorrelated across products. In the following subsections we will consider bilateral bundling in two extreme cases of perfectly correlated valuations: positive and negative correlation, respectively. As the following analysis demonstrates, the main qualitative results obtained in our baseline model extend to these two extreme cases of correlated valuations.

5.1 Positive Correlation

Assume for all consumers that $\theta_X = \theta_Y = \theta$, with θ uniformly distributed in the interval $[0, 1]$. Assume also that $\theta_Y^{*b} > \theta_X^{*a} \Leftrightarrow \frac{P_Y - p_Y + \gamma_B}{s} > \frac{P_X - p_X - \gamma_A}{s}$ and $\theta_Y^{*a} < \theta_X^{*b} \Leftrightarrow \frac{P_Y - p_Y - \gamma_A}{s} <$

$\frac{P_X - p_X + \gamma_B}{s}$.[20] Then, no consumer will purchase products of different quality.[21] Hence,

$$\begin{aligned} Q_{A_X,B_Y} &= Q_{B_X,A_Y} = 0 \\ Q_{A_X,A_Y} &= \left(1 - \frac{P_X + P_Y - p_X - p_Y - \gamma_A + \gamma_B}{2s}\right) \\ Q_{B_X,B_Y} &= \frac{P_X + P_Y - p_X - p_Y - \gamma_A + \gamma_B}{2s} \end{aligned}$$

Equilibrium prices that result from the maximization of the firms' objective functions are

$$\begin{aligned} P_X(\gamma_A, \alpha_A) &= P_Y(\gamma_A, 1-\alpha_A) = \frac{6}{5}s + \alpha_A\gamma_A \\ p_X(\gamma_B, \alpha_B) &= p_Y(\gamma_B, 1-\alpha_B) = \frac{4}{5}s + \alpha_B\gamma_B \end{aligned}$$

and the corresponding quantities are

$$Q_{A_X,A_Y} = \frac{3}{5} \text{ and } Q_{B_X,B_Y} = \frac{2}{5}$$

Note that the individual profit functions are independent of the discount and of the discount sharing rule. This is because the discount cancels out as prices equal the discount plus a constant. This results from the fact each pair of partner firms has three instruments, the two headline prices and the discount, to determine how much each partner firm will receive from selling the bundle. The equilibrium profits are

$$\begin{aligned} \Pi_{A_X} &= \Pi_{A_Y} = \left(\frac{6}{5}s + \gamma_A\alpha_A - \gamma_A\alpha_A\right)\frac{3}{5} = \frac{18}{25}s \\ \Pi_{B_X} &= \Pi_{B_Y} = \left(\frac{4}{5}s + \alpha_B\gamma_B - \gamma_B\alpha_B\right)\frac{2}{5} = \frac{8}{25}s \end{aligned}$$

Consumer surplus and welfare are given by

$$\begin{aligned} CS &= \int_0^{\frac{2}{5}} \left(\theta 2s_B - \frac{8}{5}s\right) d\theta + \int_{\frac{2}{5}}^{1} \left(\theta 2s_A - \frac{12}{5}s\right) d\theta = \frac{56 - 31\frac{s_A}{s_B}}{25} \\ W &= \int_0^{\frac{2}{5}} (\theta 2s_B) d\theta + \int_{\frac{2}{5}}^{1} (\theta 2s_A)\, d\theta = \frac{21\frac{s_A}{s_B} + 4}{25} \end{aligned}$$

It should be noted that the conditions on the θ's are verified ex-post, for any pair of positive discounts.[22] The following results still hold, when compared to the case of no discount:

[20] Note that these conditions cannot hold simultaneously if $\gamma_A = \gamma_B = 0$. Therefore, the no discount case cannot be obtained as a special case of what follows.

[21] This follows from the fact that there is no value for θ such that:
$\theta(s_A+s_B)-P_X-p_Y > \theta(s_B+s_B)-p_X-p_Y+\gamma_B$ and $\theta(s_A+s_B)-P_X-p_Y > \theta(s_A+s_A)-P_X-P_Y+\gamma_A$. This can be simplified to $\theta > \theta_X^{*b} := \frac{P_X - p_X + \gamma_B}{s}$ and $\theta < \theta_Y^{*a} := \frac{P_Y - p_Y - \gamma_A}{s}$, which is impossible since $\theta_Y^{*a} < \theta_X^{*b}$ (and likewise for the case of high quality Y and low quality X).

[22] The conditions are: $\theta_Y^{*b} > \theta_X^{*a} \Leftrightarrow \gamma_A(1-\alpha_A) + \alpha_B\gamma_B > 0$ and $\theta_Y^{*a} < \theta_X^{*b} \Leftrightarrow (\gamma_B(\alpha_B - 1) - \alpha_A\gamma_A) < 0$. In appendix B we show that there is no equilibrium if these conditions are not fulfilled.

i) the price of the high quality bundle, net of the discount, will rise.

ii) the price of the low quality bundle, net of the discount, will rise.

iii) consumer surplus and welfare will fall.

5.2 Negative Correlation

Assume for all consumers that $\theta_X = 1 - \theta_Y$ with θ_Y uniformly distributed in the interval $[0, 1]$.

Suppose that $\theta_Y^{*b} + \theta_X^{*a} = \theta_Y^{*a} + \theta_X^{*b} = 1$ which is equivalent to $s = P_X + P_Y - \gamma_A - (p_Y + p_X - \gamma_B)$. In this case, consumers will be indifferent between the high quality bundle and the low quality bundle.[23] We assume that each consumer will randomly choose one of the bundles with equal probability. Then

$$
\begin{aligned}
Q_{A_X,B_Y} &= \frac{P_Y - p_Y - \gamma_A}{s} \\
Q_{B_X,A_Y} &= \frac{P_X - p_X - \gamma_A}{s} \\
Q_{A_X,A_Y} &= \frac{1}{2}\left(1 - \frac{P_Y - p_Y - \gamma_A}{s} - \frac{P_X - p_X - \gamma_A}{s}\right) \\
Q_{B_X,B_Y} &= \frac{1}{2}\left(1 - \frac{P_Y - p_Y - \gamma_A}{s} - \frac{P_X - p_X - \gamma_A}{s}\right)
\end{aligned}
$$

Equilibrium prices that result from the maximization of the firms' objective functions are:

$$
\begin{aligned}
P_X &= s + \frac{1}{5}\gamma_A + \frac{1}{5}\gamma_B + \frac{3}{5}\alpha_A\gamma_A - \frac{2}{5}\alpha_B\gamma_B \\
P_Y &= s + \frac{4}{5}\gamma_A - \frac{1}{5}\gamma_B - \frac{3}{5}\alpha_A\gamma_A + \frac{2}{5}\alpha_B\gamma_B \\
p_X &= s - \frac{1}{5}\gamma_A - \frac{1}{5}\gamma_B + \frac{2}{5}\alpha_A\gamma_A - \frac{3}{5}\alpha_B\gamma_B \\
p_Y &= s + \frac{1}{5}\gamma_A - \frac{4}{5}\gamma_B - \frac{2}{5}\alpha_A\gamma_A + \frac{3}{5}\alpha_B\gamma_B
\end{aligned}
$$

In addition, aggregate profits are

$$
\begin{aligned}
\Pi_A &= \frac{(12\gamma_A\gamma_B + \alpha_A\gamma_A\gamma_B + \alpha_B\gamma_A\gamma_B - 2\alpha_A\alpha_B\gamma_A\gamma_B + 25s^2 - 14\gamma_A^2 + \gamma_B^2 + 6\alpha_A\gamma_A^2 - 4\alpha_B\gamma_B^2 - 6\alpha_A^2\gamma_A^2}{25s} \\
\Pi_B &= \frac{-(25s\gamma_B + 13\gamma_A\gamma_B - \alpha_A\gamma_A\gamma_B - \alpha_B\gamma_A\gamma_B + 2\alpha_A\alpha_B\gamma_A\gamma_B - 25s^2 - \gamma_A^2 - 11\gamma_B^2 + 4\alpha_A\gamma_A^2 - 6\alpha_B}{25s}
\end{aligned}
$$

Maximization of Π_A and Π_B with respect to α_A and α_B yields $\alpha_A = \alpha_B = \frac{1}{2}$. In order to have $1 - \frac{P_X - p_X - \gamma_A}{s} = \frac{P_Y - p_Y + \gamma_B}{s}$ we need that $\gamma_B = \frac{s}{2}$ or $\beta_B = \frac{1}{2}$. At $\alpha_A = \alpha_B = \beta_B = \frac{1}{2}$,

[23] In appendix B we show that there is no equilibrium if this condition is not fulfilled.

aggregate profits are then

$$\Pi_A = \left(-\frac{1}{4}\right)\left(2\beta_A^2 - \beta_A - 4\right)s$$
$$\Pi_B = \left(-\frac{1}{8}\right)\left(2\beta_A - 5\right)s,$$

from where we obtain that the optimal high quality discount is such that $\beta_A = \frac{1}{4}$. The corresponding prices are

$$P_X = P_Y = \frac{9}{8}s \text{ and } p_X = p_Y = \frac{3}{4}s$$

with

$$\theta_Y^{*a} = \theta_X^{*a} = 2^{-3} \in [0,1] \text{ and } \theta_Y^{*b} = \theta_X^{*b} = 7 \times 2^{-3} \in [0,1]$$

As $-\frac{1}{8}\left(2 * \frac{1}{4} - 5\right)s > \frac{1}{18}\left(2 - \beta_A\right)^2 s$ for any β_A, the low quality firms will always prefer to set $\beta_B = \frac{1}{2}$.

Consumer surplus and welfare are given by

$$\begin{aligned}
CS &= \int_0^{\frac{1}{8}} \left(\theta(s_B) + (1-\theta)s_A - \left(\frac{9}{8} + \frac{3}{4}\right)(s_A - s_B)\right) d\theta + \\
&\quad + \int_{\frac{1}{8}}^{\frac{7}{8}} \left(s_A - \left(2 * \frac{9}{8} - \frac{1}{4}\right)(s_A - s_B)\right) d\theta + \int_{\frac{7}{8}}^{1} \left(\theta(s_A) + (1-\theta)s_B - \left(\frac{9}{8} + \frac{3}{4}\right)(s_A - s_B)\right) d\theta \\
&= \frac{1}{64}\left(127 s_B - 63 s_A\right) = \frac{1}{64}\left(127 - 63\frac{s_A}{s_B}\right) \\
W &= \int_0^{\frac{1}{8}} \left(\theta(s_B) + (1-\theta)s_A\right) d\theta + \int_{\frac{1}{8}}^{\frac{7}{8}} (s_A)\, d\theta + \int_{\frac{7}{8}}^{1} \left(\theta(s_A) + (1-\theta)s_B\right) d\theta \\
&= \frac{1}{64}\left(63 s_A + s_B\right) = \frac{1}{64}\left(63\frac{s_A}{s_B} + 1\right)
\end{aligned}$$

As in the case of positive correlation, the following results still hold, when compared to the case of no discount:

i) the price of the high quality bundle, net of the discount, will rise.

ii) the price of the low quality bundle, net of the discount, will rise.

iii) consumer surplus will fall.[24]

6 Conclusion

Bundled discounts provide the purchasers the opportunity to pay less for a bundle of products than if they purchased each item in the package separately at the corresponding

[24] Total welfare will, however, increase.

headline price. Despite the fact that this business practice is ubiquitous in today's society, economic theory has devoted very scarce attention to this issue until recently.

The present paper studies the consequences of bundled discounts in an oligopoly setting where pairs of firms sell vertically differentiated and otherwise unrelated products. More specifically, we investigate the effects induced by the introduction of bundled discounts under three different scenarios: (*i*) unilateral bundled discount by the low quality firms; (*ii*) unilateral bundled discount by the high quality firms; and (*iii*) bilateral bundling.

Some interesting results are obtained regarding the competitive effects of bundled discounts, allowing to shed some light on the understanding of the potential antitrust risks associated with this particular type of discount arrangements. First, whenever bundled discounts are offered by (one or two pairs of) firms, then, relative to the no-discounting benchmark case, the headline prices of the bundling firms always increase whereas the headline prices of the firms not involved in discounting (if any) decrease. Second, in none of the three studied scenarios should bundled discounts be free of antitrust concerns: bundled discounts always induce a decrease both in consumer surplus and in total welfare. Third, when firms make simultaneous decisions regarding their eventual participation in a bundled discount scheme, it turns out that offering a bundled discount is a dominant strategy both for the high quality firms and for the low quality firms. In addition, in this bilateral bundling (Nash) equilibrium, and relative to the no-discounting benchmark case: (*i*) both the price of the high quality bundle and that of the low quality bundle, net of the corresponding discount, rise; and (*ii*) all firms earn higher profits. Furthermore, the equilibrium with bilateral bundled discounts corresponds to the worst scenario in terms of both consumer surplus and social welfare. This then suggests that competition authorities should scrutinize in detail the use of bundled discounts in industries where suppliers offer vertically differentiated products.

Appendix A

Proof of Lemma 1: Assume initially that there are no discounts. Consumers purchase $i = X, Y$ from A_i if and only if

$$V + \theta_i s_A - P_i > V + \theta_i s_B - p_i \Leftrightarrow \theta_i > \theta_i^* := \frac{P_i - p_i}{s}$$

Assume now that the high quality firms introduce a discount γ_A. Then:
If $\theta_X > \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will still purchase from A_X, A_Y.
If $\theta_X > \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y < \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase from A_X, A_Y if

$$\theta_X s_A - P_X + \theta_Y s_A - P_Y + \gamma_A > \theta_X s_A - P_X + \theta_Y s_B - p_Y \Leftrightarrow \theta_Y > \theta_Y^{*a} := \frac{P_Y - p_Y - \gamma_A}{s}$$

If $\theta_X < \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y < \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase from A_X, A_Y if

$$\theta_X s_A - P_X + \theta_Y s_A - P_Y + \gamma_A > \theta_X s_B - p_X + \theta_Y s_B - p_Y \Leftrightarrow \theta_X > \frac{P_X - p_X + P_Y - p_Y - \gamma_A}{s} - \theta_Y$$

If $\theta_X < \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase from A_X, A_Y if

$$\theta_X s_A - P_X + \theta_Y s_A - P_Y + \gamma_A > \theta_X s_B - p_X + \theta_Y s_A - P_Y \Leftrightarrow \theta_X > \theta_X^{*a} = \frac{P_X - p_X - \gamma_A}{s}$$

Assume now that the low quality firms also introduce a discount, γ_B. Then:

If $\theta_X < \theta_X^{*a}$ and $\theta_Y > \theta_Y^*$ will purchase from B_X, B_Y if

$$\theta_X s_B - p_X + \theta_Y s_B - p_Y + \gamma_B > \theta_X s_B - p_X + \theta_Y s_A - P_Y \Leftrightarrow \theta_Y < \theta_Y^{*b} := \frac{P_Y - p_Y + \gamma_B}{s}$$

If $\theta_X > \theta_X^*$ and $\theta_Y < \theta_Y^{*a}$ will purchase from B_X, B_Y if

$$\theta_X s_B - p_X + \theta_Y s_B - p_Y + \gamma_B > \theta_X s_A - P_X + \theta_Y s_B - p_Y \Leftrightarrow \theta_X < \theta_X^{*b} := \frac{P_X - p_X + \gamma_B}{s}$$

If $\theta_X > \theta_X^{*a}$ and $\theta_Y > \theta_Y^{*a}$ and $\theta_X > \frac{P_X - p_X + P_Y - p_Y - \gamma_A}{s} - \theta_Y$ will purchase from B_X, B_Y if

$$\theta_X s_B - p_X + \theta_Y s_B - p_Y + \gamma_B > \theta_X s_A - P_X + \theta_Y s_A - P_Y + \gamma_A$$

which is equivalent to

$$\theta_X < \frac{P_X - p_X + P_Y - p_Y + \gamma_B - \gamma_A}{s} - \theta_Y = \theta_X^{*a} + \theta_Y^{*b} - \theta_Y.$$

The demand functions result from calculating the relevant areas in the (θ_X, θ_Y) −space with $(\theta_X, \theta_Y) \in [0,1]^2$. ■

Proof of Proposition 1: Parts (i) to (iii) follow directly from the text. As for (iv), aggregate consumer surplus, CS_1, is given by the following expression

$$\int_{\theta_Y^{*b}}^{1}\int_{0}^{\theta_X^{*a}} (\theta_X s_B + \theta_Y s_A - p_X - P_Y)\, d\theta_X d\theta_Y + \int_{0}^{\theta_Y^{*a}}\int_{\theta_X^{*b}}^{1} (\theta_X s_A + \theta_Y s_B - P_X - p_Y)\, d\theta_X d\theta_Y +$$
$$\int_{0}^{\theta_Y^{*a}}\int_{0}^{\theta_X^{*b}} (s_B(\theta_X + \theta_Y) - p_X - p_Y)\, d\theta_X d\theta_Y + \int_{\theta_Y^{*a}}^{\theta_Y^{*b}}\int_{0}^{\theta_X^{*b}+\theta_Y^{*a}-\theta_Y} (s_B(\theta_X + \theta_Y) - p_X - p_Y)\, d\theta_X d\theta_Y +$$
$$\int_{\theta_Y^{*b}}^{1}\int_{\theta_X^{*a}}^{1} (s_A(\theta_X + \theta_Y) - P_X - P_Y)\, d\theta_X d\theta_Y + \int_{\theta_Y^{*a}}^{\theta_Y^{*b}}\int_{\theta_X^{*b}+\theta_Y^{*a}-\theta_Y}^{1} (s_A(\theta_X + \theta_Y) - P_X - P_Y)\, d\theta_X d\theta_Y,$$

from where we obtain

$$CS_1 = \frac{6.757\,8 - 1.319\,3\frac{s_A}{s_B}}{5.4385}$$

So,

$$CS_1 - CS_0 = \frac{6.757\,8 - 1.319\,3\frac{s_A}{s_B}}{5.4385} - \left(11 - 2\frac{s_A}{s_B}\right)/9 = \left(-\frac{9967}{489\,465}\right)\left(\frac{s_A}{s_B} - 1\right) < 0$$

Finally, aggregate welfare is given by

$$W_1 = \frac{8.567\,8\frac{s_A}{s_B} + 1.1658}{9.7335}$$

This is obtained merely by removing the prices in the expression for consumer surplus. So,

$$W_1 - W_0 = \frac{8.5678\frac{s_A}{s_B} + 1.1658}{9.7335} - (8\frac{s_A}{s_B} + 1)/9 = \left(-\frac{842}{97\,335}\right)\left(\frac{s_A}{s_B} - 1\right) < 0$$

This completes the proof. ■

Proof of Proposition 2: Parts (i) to (iii) follow directly from the text. As for (iv), aggregate consumer surplus, CS_2, is given by the same expression as in Proposition 1 with the obvious substitution of the corresponding equilibrium prices and $\theta_Y^{*a}, \theta_X^{*a}, \theta_Y^{*b}, \theta_X^{*b}$'s.

$$CS_2 = \frac{22\,444\,673 - 8795\,606\frac{s_A}{s_B}}{13\,649\,067}$$

So,

$$CS_2 - CS_0 = \frac{22\,444\,673 - 8795\,606\frac{s_A}{s_B}}{13\,649\,067} - \left(11 - 2\frac{s_A}{s_B}\right)/9 = \left(-\frac{5762\,480}{13\,649\,067}\right)\left(\frac{s_A}{s_B} - 1\right) < 0.$$

Finally, aggregate welfare is given by

$$W_2 = \frac{11\,819\,119\frac{s_A}{s_B} + 1829\,948}{13\,649\,067}$$

So,

$$W_2 - W_0 = \frac{11\,819\,119\frac{s_A}{s_B} + 1829\,948}{13\,649\,067} - (8\frac{s_A}{s_B} + 1)/9 = \left(-\frac{313\,385}{13\,649\,067}\right)\left(\frac{s_A}{s_B} - 1\right) < 0$$

This completes the proof. ■

Proof of Proposition 3: Parts (i) to (iv) follow directly from the text. As for (v), aggregate consumer surplus, CS_3, is given by

$$
\begin{aligned}
CS_3 &= \int_0^{\frac{\sqrt{6}}{3}} \left(\int_0^{\frac{\sqrt{6}}{3}-\theta_Y} \left(s_B \left(\theta_X + \theta_Y \right) - p_X - p_Y + \gamma_B \right) d\theta_X \right) d\theta_Y + \\
&+ \int_{\frac{\sqrt{6}}{3}}^1 \left(\int_0^1 \left(s_A \left(\theta_X + \theta_Y \right) - P_X - P_Y + \gamma_A \right) d\theta_X \right) d\theta_Y + \\
&+ \int_0^{\frac{\sqrt{6}}{3}} \left(\int_{\frac{\sqrt{6}}{3}-\theta_Y}^1 \left(s_A \left(\theta_X + \theta_Y \right) - P_X - P_Y + \gamma_A \right) d\theta_X \right) d\theta_Y \\
&= \frac{1}{27} \left(17\sqrt{6} + \frac{s_A}{s_B} \left(27 - 17\sqrt{6} \right) \right)
\end{aligned}
$$

So,

$$
CS_3 - CS_0 = \frac{1}{27} \left(17\sqrt{6} + \frac{s_A}{s_B} \left(27 - 17\sqrt{6} \right) \right) - \left(11 - 2\frac{s_A}{s_B} \right) /9 = \frac{17\sqrt{6} - 33}{27} \left(1 - \frac{s_A}{s_B} \right) < 0.
$$

Finally, aggregate welfare is given by

$$
W_3 = \frac{1}{27} \left(2\sqrt{6} + \frac{s_A}{s_B} \left(27 - 2\sqrt{6} \right) \right)
$$

So,

$$
W_2 - W_0 = \frac{1}{27} \left(2\sqrt{6} + \frac{s_A}{s_B} \left(27 - 2\sqrt{6} \right) \right) - (8\frac{s_A}{s_B} + 1)/9 = \frac{3 - 2\sqrt{6}}{27} \left(\frac{s_A}{s_B} - 1 \right) < 0
$$

This completes the proof. ■

Appendix B

In this appendix we address the cases left out of the analysis in Section 5.

Positive Correlation

Assume first that $\theta_Y^{*a} > \theta_X^{*b} \Leftrightarrow \frac{P_Y - p_Y - \gamma_A}{s} > \frac{P_X - p_X + \gamma_B}{s}$. Then

$$
\begin{aligned}
Q_{A_X,B_Y} &= \frac{P_Y - p_Y - \gamma_A}{s} - \frac{P_X - p_X + \gamma_B}{s} \\
Q_{B_X,A_Y} &= 0 \\
Q_{A_X,A_Y} &= \left(1 - \frac{P_Y - p_Y - \gamma_A}{s} \right) \\
Q_{B_X,B_Y} &= \frac{P_X - p_X + \gamma_B}{s}
\end{aligned}
$$

Equilibrium prices result from the maximization of the following objective functions:

$$
\begin{aligned}
\Pi_{A_X} &= P_X\left(Q_{A_X,B_Y}+Q_{A_X,A_Y}\right)-\gamma_A\alpha_A Q_{A_X,A_Y}\\
\Pi_{A_Y} &= P_Y\left(Q_{A_X,A_Y}\right)-\gamma_A\left(1-\alpha_A\right)Q_{A_X,A_Y}\\
\Pi_{B_X} &= p_X\left(Q_{B_X,B_Y}\right)-\gamma_B\alpha_B Q_{B_X,B_Y}\\
\Pi_{B_Y} &= p_Y\left(Q_{A_X,B_Y}+Q_{B_X,B_Y}\right)-\gamma_B\left(1-\alpha_B\right)Q_{B_X,B_Y}
\end{aligned}
$$

yielding

$$
\begin{aligned}
P_X &= \left(\frac{2}{3}s-\frac{1}{3}\gamma_B+\frac{1}{3}\alpha_B\gamma_B\right) \text{ and } P_Y=\left(\frac{2}{3}s+\gamma_A-\frac{2}{3}\alpha_A\gamma_A\right)\\
p_X &= \left(\frac{1}{3}s+\frac{1}{3}\gamma_B+\frac{2}{3}\alpha_B\gamma_B\right) \text{ and } p_Y=\left(\frac{1}{3}s-\frac{1}{3}\alpha_A\gamma_A\right)
\end{aligned}
$$

Note that it is impossible to have $\theta_Y^{*a}>\theta_X^{*b}\Leftrightarrow\frac{P_Y-p_Y-\gamma_A}{s}>\frac{P_X-p_X+\gamma_B}{s}\Leftrightarrow-\gamma_B\left(1-\alpha_B\right)-\alpha_A\gamma_A>0$.

Assume now that $\theta_Y^{*b}<\theta_X^{*a}\Leftrightarrow\frac{P_Y-p_Y+\gamma_B}{s}<\frac{P_X-p_X-\gamma_A}{s}$. Then,

$$
\begin{aligned}
Q_{A_X,B_Y} &= 0\\
Q_{B_X,A_Y} &= \frac{P_X-p_X-\gamma_A}{s}-\frac{P_Y-p_Y+\gamma_B}{s}\\
Q_{A_X,A_Y} &= 1-\frac{P_X-p_X-\gamma_A}{s}\\
Q_{B_X,B_Y} &= \frac{P_Y-p_Y+\gamma_B}{s}
\end{aligned}
$$

Equilibrium prices result from the maximization of the following objective functions:

$$
\begin{aligned}
\Pi_{A_X} &= P_X\left(Q_{A_X,A_Y}\right)-\gamma_A\alpha_A Q_{A_X,A_Y}\\
\Pi_{A_Y} &= P_Y\left(Q_{A_X,A_Y}+Q_{B_X,A_Y}\right)-\gamma_A\left(1-\alpha_A\right)Q_{A_X,A_Y}\\
\Pi_{B_X} &= p_X\left(Q_{B_X,B_Y}+Q_{B_X,A_Y}\right)-\gamma_B\alpha_B Q_{B_X,B_Y}\\
\Pi_{B_Y} &= p_Y\left(Q_{B_X,B_Y}\right)-\gamma_B\left(1-\alpha_B\right)Q_{B_X,B_Y}
\end{aligned}
$$

yielding

$$
\begin{aligned}
P_X &= \left(\frac{2}{3}s+\frac{1}{3}\gamma_A+\frac{2}{3}\alpha_A\gamma_A\right) \text{ and } P_Y=\left(\frac{2}{3}s-\frac{1}{3}\alpha_B\gamma_B\right)\\
p_X &= \left(\frac{1}{3}s-\frac{1}{3}\gamma_A+\frac{1}{3}\alpha_A\gamma_A\right) \text{ and } p_Y=\left(\frac{1}{3}s+\gamma_B-\frac{2}{3}\alpha_B\gamma_B\right)
\end{aligned}
$$

Note that it is impossible to have $\theta_Y^{*b}<\theta_X^{*a}\Leftrightarrow\frac{P_Y-p_Y+\gamma_B}{s}<\frac{P_X-p_X-\gamma_A}{s}\Leftrightarrow\left(1-\alpha_A\right)\gamma_A+\alpha_B\gamma_B<0$.

Negative Correlation

Assume first that $\theta_Y^{*b}+\theta_X^{*a}=\theta_Y^{*a}+\theta_X^{*b}=\frac{P_X+P_Y-p_X-p_Y-\gamma_A+\gamma_B}{s}>1$. Then, the high quality bundle will not be sold and demand is given by

$$
\begin{aligned}
Q_{A_X,B_Y} &= 1-\frac{P_X-p_X+\gamma_B}{s} \\
Q_{B_X,A_Y} &= 1-\frac{P_Y-p_Y+\gamma_B}{s} \\
Q_{A_X,A_Y} &= 0 \\
Q_{B_X,B_Y} &= 1-\left(1-\frac{P_X-p_X+\gamma_B}{s}\right)-\left(1-\frac{P_Y-p_Y+\gamma_B}{s}\right)
\end{aligned}
$$

Equilibrium prices that result from the maximization of the firms' objective functions are:

$$
\begin{aligned}
P_X(\gamma_B,\alpha_B) &= P_Y(\gamma_B,\alpha_B)=\left(\frac{2}{3}s-\frac{1}{3}\gamma_B+\frac{1}{3}\alpha_B\gamma_B\right) \\
p_X(\gamma_B,\alpha_B) &= p_Y(\gamma_B,\alpha_B)=\left(\frac{1}{3}s+\frac{1}{3}\gamma_B+\frac{2}{3}\alpha_B\gamma_B\right)
\end{aligned}
$$

Note that prices and quantities are independent of γ_A because there is no demand for the low quality bundle.

Maximizing the joint profits of the high quality firms evaluated at the equilibrium prices, with respect to α_B gives the relevant solution at $\alpha_B=\frac{1}{2}$.[25] Note that $\Pi_B(\frac{1}{2})=\left(2-\beta_B^2+8\beta_B\right)s/9$ is increasing in β_B for the admissible range of discounts and that the condition $\theta_Y^{*b}+\theta_X^{*a}=\theta_Y^{*a}+\theta_X^{*b}=\frac{P_X+P_Y-p_X-p_Y-\gamma_A+\gamma_B}{s}>1$ is not verified ex-post: $\theta_Y^{*b}+\theta_X^{*a}=\theta_Y^{*a}+\theta_X^{*b}=\frac{(P_X+P_Y-p_X-p_Y-\gamma_A+\gamma_B)}{s}=\left(-\frac{1}{3}\right)\left(3\beta_A+2\beta_B-2\right)=\frac{2}{3}-\frac{2}{3}\beta_B-\beta_A<1$.

Assume now that $\theta_Y^{*b}+\theta_X^{*a}=\theta_Y^{*a}+\theta_X^{*b}=\frac{P_X+P_Y-p_X-p_Y-\gamma_A+\gamma_B}{s}<1$. Then there will be no consumer purchasing the low quality bundle. Demand functions are

$$
\begin{aligned}
Q_{A_X,B_Y} &= \frac{P_Y-p_Y-\gamma_A}{s} \\
Q_{B_X,A_Y} &= \frac{P_X-p_X-\gamma_A}{s} \\
Q_{A_X,A_Y} &= \left(1-\frac{P_X-p_X-\gamma_A}{s}-\frac{P_Y-p_Y-\gamma_A}{s}\right) \\
Q_{B_X,B_Y} &= 0
\end{aligned}
$$

Equilibrium prices that result from the maximization of the firms' objective functions are

$$
\begin{aligned}
P_X(\gamma_A,\alpha_A) &= P_Y(\gamma_A,1-\alpha_A)=\left(\frac{2}{3}s+\frac{1}{3}\gamma_A+\frac{2}{3}\alpha_A\gamma_A\right) \\
p_X(\gamma_A,\alpha_A) &= p_Y(\gamma_A,1-\alpha_A)=\left(\frac{1}{3}s-\frac{1}{3}\gamma_A+\frac{1}{3}\alpha_A\gamma_A\right)
\end{aligned}
$$

[25] This is the unique solution to $\partial\Pi_B/\partial\alpha_B=0$ for any positive γ_B. Second-order conditions are always verified.

Note that prices and quantities are independent of γ_B because there is no demand for the low quality bundle.

Maximizing the joint profits of the high quality firms, $\Pi_A(\alpha_A) := \Pi_{A_X} + \Pi_{A_Y}$ evaluated at the equilibrium prices, with respect to α_A gives the relevant solution at $\alpha_A = \frac{1}{2}$.[26] The optimal discount is obtained by maximizing joint profit:

$$\begin{aligned} \Pi_A(\frac{1}{2}) &= \frac{1}{9}(8-\beta_A)(\beta_A+1)\,s. \\ \Pi_B(\frac{1}{2}) &= \frac{1}{18}(2-\beta_A)^2\,s \end{aligned}$$

Note that $\Pi_A(\frac{1}{2})$ is increasing in γ_A for the admissible range of discounts and such that $\frac{(P_X+P_Y-p_X-p_Y-\gamma_A+\gamma_B)}{s} = \frac{1}{3}s^{-1}(2s+2s\beta_A+3s\beta_B) = \frac{2}{3}\beta_A+\beta_B+\frac{2}{3} < 1 \Leftrightarrow \beta_A < \frac{1}{2}-\frac{3}{2}\beta_B$. Therefore, β_A will increase until $\frac{(P_X+P_Y-p_X-p_Y-\gamma_A+\gamma_B)}{s} \to 1$.

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